Quantum Gauge Invariance and Grand Unification

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Zusammenfassung

In dieser Arbeit wird gezeigt, wie Quanteneichinvarianz Grosse Vereinheitlichte Theorien, insbesondere das von Georgi und Glashow vorgeschlagene $SU(5)$ Modell, einschränkt. Dies geschieht im Rahmen der kausalen Störungstheorie nach Epstein und Glaser. Im ersten Teil der Arbeit werden die nötigen Eichbedingungen aus erster und zweiter Ordnung zusammengestellt; es ergeben sich Beziehungen zwischen den Yang-Mills Strukturkonstanten und den Massen der Eichbosonen. Diese werden im zweiten Teil in einem ersten Schritt direkt mit konkreten Strukturkonstanten auf $SU(5)$ angewandt. In einem zweiten Schritt werden keine konkreten Werte mehr verwendet, sondern die Strukturkonstanten nur durch Ladungserhaltung gemäss $SU(5)$ und Quanteneichinvarianz eingeschränkt. In beiden Schritten zeigt sich: Quanteneichinvarianz und $SU(5)$ sind nicht kompatibel. In einem dritten Schritt werden die Strukturkonstanten (abgesehen von physikalischen und technischen Annahmen) frei gelassen; das sich ergebende Gleichungssystem ist linear in den Produkten der Strukturkonstanten. Für diese Produkte existieren Lösungen, so dass die Theorie jetzt eichinvariant ist. Diese Lösungen werden für eine Theorie mit drei und vier Bosonenmassen ausgewertet und auf ihre Grössenordnungen hin untersucht. Es zeigt sich, dass es keine Lösung gibt, in der alle Produkte von Ordnung eins sind.

Abstract

In this work it is shown in the framework of causal perturbation theory, how Quantum Gauge Invariance gives restrictions for grand unified models, particularly Georgi-Glashow $SU(5)$. In the first part the necessary gauge restrictions from first and second order are assembled. In second order there are relations between the Yang-Mills structure constants and the masses of the gauge bosons. These relations are reconsidered in the second part in the case of $SU(5)$: In a first step a check with concrete values for the structure constants is executed, in a second step the structure constants are only restricted by charge conservation according to $SU(5)$ and gauge invariance. In both steps one sees: Quantum Gauge Invariance and $SU(5)$ are not compatible. In a third step the structure constants are left free (apart from physical and technical assumptions). The resulting system of equations is linear in the products of structure constants. It has non-trivial solutions so that the theory now satisfies gauge invariance. These solutions are analysed for a theory with three and four gauge boson masses with regard to their orders of magnitude. It appears that there is no solution with all products of order one.

Natur und Kunst, sie scheinen sich zu fliehen
Und haben sich, eh' man es denkt, gefunden;
Der Widerwille ist auch mir verschwunden,
Und beide scheinen gleich mich anzuziehen.

Es gilt wohl nur ein redliches Bemühen!
Und wenn wir erst in abgemeßnen Stunden
Mit Geist und Fleiß uns an die Kunst gebunden,
Mag frei Natur im Herzen wieder glühen.

So ist's mit aller Bildung auch beschaffen:
Vergebens werden ungebundne Geister
Nach der Vollendung reiner Höhe streben.

Wer Großes will, muß sich zusammenraffen;
In der Beschränkung zeigt sich erst der Meister,
Und das Gesetz nur kann uns Freiheit geben.

Johann Wolfgang von Goethe, um 1800

Contents

Part 1. Restrictions from perturbative gauge invariance

1. Introduction

As the name suggests, grand unification is a capacious undertaking. Although predictions of the Standard Model are excellently confirmed by experiment, there remain various open questions, among them the mass spectrum of the elementary particles. Grand unified models provide convincing answers to many of these problems and are therefore greatly favoured. In one of the simplest, $SU(5)$, the Standard Model is naturally embedded and the quark masses are roughly determined. But the neutrini remain massless and it seems inevitable to go over to bigger unification groups, as $SO(10)$, E_6, even up to E_8, containing $SU(5)$, if one wants to maintain its achievements. The alternative is to consider other unification schemes that are not fixed on simple groups and allow, as products, a composition of miscellaneous groups. However, in such a case one deals with far more free parameters that have to be explained.

It comes as no surprise that researches thus focused on the explanation of those models and their applications (and/or modifications). As a consequence the fundamentals are no more questioned. This could be reasonable though, since some of the groundwork is introduced in an heuristic way; Higgs potentials for instance are normally put in by hand and the corresponding mechanism of providing masses for the gauge bosons, is based on semi-classical arguments. One cannot be sure whether finally all works perfectly because there are more fundamental principles to be satisfied, such as gauge invariance. Therefore, we think, it is promising to address the problem from a different point of view.

In our approach we directly start at the quantum level. By means of perturbative quantum gauge invariance [we speak of *gauge invariance* for short] we look for restrictions on grand unified models. In a recent monograph [2] and related papers quantum gauge theories have already been extensively studied and it is shown, that the Glashow Weinberg Salam Model of electroweak interaction can entirely be deduced by the first three orders of gauge invariance, assuming three massive and one massless gauge field. The Higgs potential is fully determined in third order if one assumes one Higgs field. But the discussion of gauge invariance is even more general: In second order one finds for a very general massive gauge theory, restrictions on the masses of the gauge fields and the structure constants of the theory. That means, one can explicitly check whether other gauge theories than the Standard Model or theories beyond the Standard Model are gauge invariant.

Recent results by Dütsch and Fredenhagen [21], [22] imply the equivalence between perturbative quantum gauge invariance and BRST invariance. There are indeed some obvious similarities between our approach and BRST. Yet we work throughout with free asymptotic fields in the framework of S-matrix theory which is conceptually easier and avoids many mathematical difficulties.

After a brief summary of quantum gauge invariance, where the restrictions for a general massive gauge theory are given, we focus on $SU(5)$ by a short review of the Georgi-Glashow model. Most important for our examination is the assignment of the masses and charges to the gauge bosons in that theory. Having this knowledge in mind, we look at which couplings and thus which structure constants can be different from zero. This can in principle be done in two ways: by means of gauge invariance or by charge and colour conservation. We examine all gauge restrictions with the mass degeneracy of the Georgi-Glashow model and the structure constants that are not zero. This leads to a system of equations that is very restrictive and does not allow for the Georgi-Glashow model in this specific setting. In the following subsections we alter the mass degeneracy for the two new sets of gauge bosons X and Y. Since this does not help, we go over to a more generic point of view and look for a model with four masses but arbitrary couplings. In this setting the contradictions to gauge invariance are removed.

2. Perturbative Quantum Gauge Invariance

We set off with a motivation for our definition of gauge invariance with the gauge variation d_Q; it is a *quantum* formulation of gauge invariance. We do not need the concept of covariant derivatives here that relies on the (classical) Lagrangian approach.

Instead we vary the vector field operators in the sense

$$A'^{\mu}(x) = A^{\mu}(x) + \lambda\partial^{\mu}u(x) + \mathcal{O}(\lambda^2). \tag{2.1}$$

$u(x)$ is a free quantum field, that fulfills the wave equation, $\Box u(x) = 0$, implying that our transformed field A' still satisfies the wave equation. Since for A' the same commutation relations should come true as for A, (2.1) has to be of the form

$$A'^{\mu}(x) = e^{-i\lambda Q}A^{\mu}(x)e^{i\lambda Q} \tag{2.2}$$

with a gauge charge Q. By using a Lie series expansion (Hausdorff-formula) it reads

$$A'^{\mu}(x) = A^{\mu}(x) + i\lambda[Q, A^{\mu}(x)] + \mathcal{O}(\lambda^2) \tag{2.3}$$

where we use square brackets to denote the commutator[1]. Now, a comparison of (2.3) with (2.1) shows

$$[Q, A^{\mu}(x)] = i\partial^{\mu}u(x) \tag{2.4}$$

which determines the gauge charge Q up to a C-number (Schur's lemma). The Q was first proposed in [28]; it can be proved that Q has the following form cf. [2], eq. (1.4.5)

$$Q = \int d^3x\, \partial_\nu A^\nu \overleftrightarrow{\partial}_0 u \tag{2.5}$$

where the double arrow indicates that the derivative ∂_0 acts on both sides

$$\partial_\nu A^\nu \overleftrightarrow{\partial}_0 u = \partial_\nu A^\nu(\partial_0 u) - (\partial_0(\partial_\nu A^\nu))u.$$

Computing the anti-commutator $\{Q, Q\}$, one sees that

$$Q^2 = 0, \tag{2.6}$$

i.e. Q is nilpotent if and only if u is quantised as a Fermi field. This is a crucial property for our gauge charge, because the physical states are in the kernel of the nilpotent operator Q.[2]

[1] For the anti-commutator we will use braces $\{\ ,\ \}$.

[2] This is the same for the Q defined in the context of BRST transformation, cf. [5], eq. (15.7.31). Yet there lies a crucial difference in the fact that in BRST interacting fields are used where we work with free fields only. For the connection of our approach to massive free BRST transformation see ([21] or [22], appendix B, where the connection to interaction theory is explained.

In the general case with F being a Wick monomial containing only Bose and an even number n_F of fermionic ghost fields u, the commutator $[Q, F]$ acts like a derivation which motivates the denotation as $[Q, F] =: d_Q$. In the case of an odd number of ghosts it is an anti-derivative $\{Q, F\}$. d_Q respects the Leibniz-rule

$$d_Q(F(x)G(y)) = (d_Q F(x))G(y) + (-1)^{n_F} F(x) d_Q G(y)$$

and the nilpotency of Q is handed down to the property

$$d_Q^2 = 0 \tag{2.7}$$

acting on Bose or Fermi fields.

Having defined gauge variation d_Q of quantum fields we turn to the chronological products T_n ("n-point function") which are expressed by asymptotic free fields, since in our approach we focus on the adiabatically switched S-matrix

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n T_n(x_1, \ldots, x_n) g(x_1) \cdots g(x_n), \tag{2.8}$$

that is "smeared out" with a test function g. In the causal approach[3] T_n are well-defined operator valued distributions, symmetric in the space coordinates $x_1, \ldots, x_n$. Chronological means that if the arguments in T_n are time ordered

$$x_1^0 > x_2^0 > \ldots > x_n^0,$$

then T_n is the ordinary product (indicating time ordering with the letter T),

$$T_n(x_1, \ldots, x_n) = \mathsf{T}[T_1(x_1) \ldots T_1(x_n)], \tag{2.10}$$

i.e. T_1 is the very basis to look at.

The gauge variation for $d_Q T_1$ can be calculated first. If

$$d_Q T_1 = i\partial_\mu T^\mu_{1/1} \tag{2.11}$$

is of divergence form, we call the theory *gauge invariant to first order*; $T^\mu_{1/1}$ is called Q-vertex, which does not mean that it describes a physical coupling but is a mathematical tool to formulate gauge invariance. The subscript $1/1$ means order one and vertex one, in general n/m stands for order n with the Q-vertex at the place x_m.

The usefulness of this definition can quickly be seen in the case of QED with the well known normally ordered

$$T_1^{\mathrm{QED}} = ie : \overline{\psi}\gamma^\mu\psi : A_\mu.$$

Since the free Fermi fields ψ, $\overline{\psi}$ have zero gauge variation $d_Q\psi = d_Q\overline{\psi} = 0$, the gauge variation for T_1^{QED} is $d_Q T_1^{\mathrm{QED}} = -e : \overline{\psi}\gamma^\mu\psi : \partial_\mu u$. Charge conservation

[3] Our S-matrix approach is referred to as *causal*; it goes back to [27] and is elaborated in [1] & [2]. In terms of the S-matrix causality for two tempered test functions g_1 and g_2 with supports

$$\operatorname{supp} g_1 \subset \{x \in \mathbb{M} | x^0 \in (-\infty, r)\},$$
$$\operatorname{supp} g_2 \subset \{x \in \mathbb{M} | x^0 \in (r, +\infty)\}$$

means

$$S(g_1 + g_2) = S(g_2)S(g_1) \tag{2.9}$$

which, translated in terms of the time ordered products T_n, constitutes the basis of causal Quantum Field Theories. The S-matrix (2.8) is in general affected by infrared and ultraviolet divergences. A carefull handling of time-ordering though, can avoid ultraviolet divergences. Infrared divergences have to be tackled in the end, performing the adiabatic limit $g \to 1$. For purely massive theories this can be achieved; with massless fields some care is needed. The existence of the limit could even constitute a selection criterion for theories realised in nature.

$\partial_\mu : \overline{\psi}\gamma^\mu\psi := 0$ implies $d_Q T_1^{\text{QED}} = ie\partial_\mu(: \overline{\psi}\gamma^\mu\psi : u)$ and the expression inside the bracket is the Q-vertex $T^\mu_{1/1} = i : \overline{\psi}\gamma^\mu\psi : u$. QED is gauge invariant to first order.

We will use the gauge variation for higher orders T_n (esp. $n = 2$ and $n = 3$) therefore we make a formal generalisation for the time ordered products

$$\begin{aligned} d_Q T_n &= d_Q \mathsf{T}\big[T_1(x_1)\cdots T_1(x_n)\big] \\ &= \sum_{l=1}^{n} \mathsf{T}\big[T_1(x_1)\cdots d_Q T_1(x_l)\cdots T_1(x_n)\big] \\ &= \sum_{l=1}^{n} \mathsf{T}\big[T_1(x_1)\cdots i\partial_\mu T^\mu_{1/1}(x_l)\cdots T_1(x_n)\big] \\ &= i\sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu}\mathsf{T}\big[T_1(x_1)\cdots T^\mu_{1/1}(x_l)\cdots T_1(x_n)\big] \\ &=: i\sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T^\mu_{n/l}(x_1,\dots,x_n). \end{aligned} \tag{2.12}$$

In this calculation we get local terms in general, due to distribution splitting ([1], sect. 3.2). If these local terms cancel each other or if we can absorb them by redifining the T-products, we call our theory *gauge invariant to nth order*. The stated definition of gauge invariance is independent of representation and one can derive the Slavnov-Taylor identities for massless gauge fields [2].

After a short presentation of the gauge theory we work with, we will state gauge conditions arising from perturbative gauge invariance.

2.1. **General massive gauge theory.** It turns out that our notion of gauge invariance is a very strong one. Starting with a most general mathematical ansatz for the gauge coupling, gauge invariance, leads to the physically relevant theories. What do we start with? From studies on gauge invariance of massless non-Abelian gauge theories it turns out that they are essentially of Yang-Mills type which means up to divergence and coboundary terms [19], [23]. Such terms lead to equivalent[4] S-matrices in the end. And second order gauge invariance of massive gauge theories necessitates the introduction of additional physical scalar fields (Higgs fields) [16]. We have therefore the following field content for the gauge fields.

We take r massive and s massless gauge fields A^μ_a, $a = 1,\dots,r+s$ together with $(r+s)$ fermionic ghost and anti-ghost fields $u_a, \tilde{u}_a$[5]. The masses of a gauge field and the corresponding ghost and anti-ghost fields are equal. We have $m_a = 0$ for $a > r$. In order to get a gauge charge Q, which is nilpotent $Q^2 = 0$, as in the massless

[4]We call two S-matrices equivalent if all matrix elements between physical states agree in the adiabatic limit:

$$\lim_{g\to 1}(\Phi, \mathsf{P}S(g)\mathsf{P}\Psi) = \lim_{g\to 1}(\Phi, \mathsf{P}S'(g)\mathsf{P}\Psi)$$

where P is a projection operator on the physical subspace $\mathcal{F}_{\text{phys}}$ and Φ and Ψ are arbitrary states in Fock space. Even stronger and more adequate for the massless case is the perturbative version

$$\mathsf{P}T_n\mathsf{P} = \mathsf{P}T'_n\mathsf{P} + \text{div}.$$

Here T_n and T'_n are n-point functions corresponding to $S(g)$ and $S'(g)$ respectively. The first definition in terms of the adiabatic limit is implied by this second stronger perturbative one.

[5]The field $\tilde{u}$ is not the adjoint field of u; this would be in conflict with the connection between spin and statistics, cf. [2], sect. 1.2

case, we have to introduce for every massive gauge vector field $A_a^\mu(x), a \le r$, a scalar partner $\Phi_a(x)$ with the same mass ${m_a}^6$, leading to

$$Q = \int d^3x \, (\partial_\nu A^\nu + m\Phi) \overleftrightarrow{\partial}_0 u,$$

as discussed in [2], sect. 1.5. Note, that the scalar and ghost fields appearing in Q are all unphysical because their excitations do not belong to the physical subspace $\mathcal{F}_{\text{phys}}$ and as in the massless case we can express this subspace as

$$\mathcal{F}_{\text{phys}} = \text{Ker}Q/\text{Ran}Q$$

or in any of its equivalent expressions (cf. [2], sect. 1.4).[7] The additional physical scalar fields (Higgs fields) we denote as $\varphi_p, p = 1, \ldots, t$ with arbitrary masses μ_p. We use indices

$$p, q, \ldots = 1, \ldots, t$$

from the end of the alphabet to number the Higgs fields, letters

$$h, j, k, l, \ldots = 1, \ldots, r$$

from the middle denote the other massive scalar fields and

$$a, b, c, d, e, f, \ldots = 1, \ldots, r + s$$

is used for the gauge fields and ghosts. These fields are quantised in the manner

$$\begin{aligned}
(\Box + m_a^2) A_a^\mu(x) &= 0 \\
[A_a^\mu(x), A_b^\nu(y)] &= i\delta_{ab} g^{\mu\nu} D_{m_a}(x-y) \\
(\Box + m_a^2) u_a(x) &= 0 = (\Box + m_a^2)\tilde{u}_a(x) \\
\{u_a(x), \tilde{u}_b(y)\} &= i\delta_{ab} D_{m_a}(x-y) \\
(\Box + m_a^2)\Phi_a(x) &= 0 \\
[\Phi_a(x), \Phi_b(y)] &= i\delta_{ab} D_{m_a}(x-y)
\end{aligned} \tag{2.13}$$

and the last two equations are also true for the physical scalar fields φ_p. D_{m_a} is the Jordan-Pauli distribution to the mass m_a [1], sect. 2.3. All Bose fields are hermitian fields. The corresponding mass matrices are already diagonal. This means one expects in the case of the electroweak theory directly to see the Z and A^μ field instead of the W^3 and the B-field (or W^0 field [7], (6.25)).

With those fields the following trilinear couplings[8] are the starting point:

$$T_1(x) = T_1^0 + T_1^1 + \ldots + T_1^{11}$$

where

$$T_1^0 = igf_{abc}(A_{\mu a} A_{\nu b} \partial^\nu A_c^\mu - A_{\mu a} u_b \partial^\mu \tilde{u}_c) \tag{2.14}$$

$$T_1^1 = igf_{ahj}^1 A_a^\mu (\Phi_h \partial_\mu \Phi_j - \Phi_j \partial_\mu \Phi_h), \; f_{ahj}^1 = -f_{ajh}^1 \tag{2.15}$$

$$T_1^2 = igf_{abh}^2 A_{\mu a} A_b^\mu \Phi_h, \; f_{abh}^2 = f_{bah}^2 \tag{2.16}$$

[6]Note that we start from the beginning with additional massive scalar fields Φ_h. We do not need something like spontaneous symmetry breaking later. For instance studying the Standard Model we can start with massive gauge bosons $W_{1,2}$ and Z. It is then clear that one sees the physically relevant "broken" gauge theory.

[7]For this reason it is convenient to call u, $\tilde{u}$ fermionic and Φ bosonic ghosts. In the old terminology one can identify the bosonic ghosts with the so called "would-be Goldstone-bosons", arising in spontaneous symmetry breaking of local symmetries.

[8]Quadrilinear couplings come out as normalisation terms for second order tree graphs.

$$T_1^3 = igf^3_{abh}\tilde{u}_a u_b \Phi_h \tag{2.17}$$

$$T_1^4 = igf^4_{hjk}\Phi_h\Phi_j\Phi_k. \tag{2.18}$$

All coupling constants are real because T_1 must be skew-adjoint. In addition the pure Yang-Mills coupling constants f_{abc} in (2.14) are totally antisymmetric, as a consequence of gauge invariance, [2], eq. (3.2.28). We have left out summation symbols here, as well as the double dots, yet all these products of field operators are normally ordered Wick products of free fields.

As mentioned, we introduce real physical (Higgs) couplings. Without them we would not achieve second order gauge invariance. We do this by a replacement of the bosonic ghosts by Higgs fields:

$$T_1^5 = igf^5_{ahp}A^\mu_a(\Phi_h\partial_\mu\varphi_p - \varphi_p\partial_\mu\Phi_h) \tag{2.19}$$

$$T_1^6 = igf^6_{apq}A^\mu_a(\varphi_p\partial_\mu\varphi_q - \varphi_q\partial_\mu\varphi_p), \quad f^6_{apq} = -f^6_{aqp} \tag{2.20}$$

$$T_1^7 = igf^7_{abp}A_{\mu a}A^\mu_b\varphi_p, \quad f^7_{abp} = f^7_{bap} \tag{2.21}$$

$$T_1^8 = igf^8_{abp}\tilde{u}_a u_b\varphi_p \tag{2.22}$$

$$T_1^9 = igf^9_{hjp}\Phi_h\Phi_j\varphi_p, \quad f^9_{hjp} = f^9_{jhp} \tag{2.23}$$

$$T_1^{10} = igf^{10}_{hpq}\Phi_h\varphi_p\varphi_q, \quad f^{10}_{hpq} = f^{10}_{hqp} \tag{2.24}$$

$$T_1^{11} = igf^{11}_{pqu}\varphi_p\varphi_q\varphi_u. \tag{2.25}$$

By first and second order gauge invariance one can relate all coupling constants to the pure Yang-Mills couplings f_{abc}. Only f^{11} remains unfixed and is determined in third order for $t = 1$. But gauge invariance not only fixes all the the coupling constants $f^1, \ldots, f^{11}$: One gets more relations between the masses of the gauge bosons and the Yang-Mills structure constants. These relations will be the basis for the forthcoming considerations. Note that this is not the most general form for the couplings. E.g. in T_1^6 the symmetric combination

$$A^\mu_a(\varphi_p\partial_\mu\varphi_q + \varphi_q\partial_\mu\varphi_p) \tag{2.26}$$

is not listed because it can be expressed by a divergence. Such terms do not change the physical S-matrix. The same is true for coboundary couplings. By not writing such terms we concentrate on the physical relevant basis of our gauge theories. Our basis is insofar fixed that we can only rotate the generators with mass degeneracy. Such a rotation is done when one combines W^1 and W^2 to the charged $W^\pm$. And of course also generators for other fields with the same mass can be rotated into one another.

2.2. **First order gauge invariance.** Let us first write down the gauge variations of the fundamental fields

$$\begin{aligned}
d_Q A^\mu_a(x) &= i\partial^\mu u_a(x)\\
d_Q\Phi_h(x) &= im_h u_h(x)\\
d_Q u_a(x) &= 0\\
d_Q\tilde{u}_a(x) &= -i(\partial_\mu A^\mu_a(x) + m_a\Phi_a(x))\\
d_Q\varphi_p(x) &= 0.
\end{aligned} \tag{2.27}$$

We now calculate the gauge variation of all terms in T_1 and transform the result to a divergence form

$$d_Q T_1 = i\partial_\mu T^\mu_{1/1}.$$

The $T^\mu_{1/1}$ appearing here is the Q-vertex. It is not unique, but a possible modification has no influence on gauge invariance of higher orders [23]. The most convenient way to achieve the desired divergence form is to take out the derivatives of the ghost fields and use the field equations. In this way we find, setting the coupling constant $g = 1$ from now on[9]

$$\begin{aligned}
d_Q T_1^0 =& + f_{abc}\Big\{\partial_\mu[A_{\nu a}u_b(\partial^\nu A^\mu_c - \partial^\mu A^\nu_c) + \frac{1}{2}u_a u_b \partial^\mu \tilde{u}_c] && (2.28)\\
&- m_c^2 A_{\nu a}u_b A^\nu_c + \frac{1}{2}m_c^2 u_a u_b \tilde{u}_c + m_c A_{\nu a} u_b \partial^\nu \Phi_c\Big\} && (2.29)\\
d_Q T_1^1 =& - f^1_{ahj}\Big\{\partial^\mu[u_a(\Phi_h\partial_\mu\Phi_j - \Phi_j\partial_\mu\Phi_h)\\
&+ m_j A^\mu_a \Phi_h u_j - m_h A^\mu_a \Phi_j u_h] + (m_j^2 - m_h^2)u_a\Phi_h\Phi_j\\
&+ m_h(\partial_\mu A^\mu_a \Phi_j + 2A^\mu_a\partial_\mu\Phi_j)u_h - m_j(\partial_\mu A^\mu_a\Phi_h + 2A^\mu_a\partial_\mu\Phi_h)u_j\Big\} && (2.30)\\
d_Q T_1^2 =& - f^2_{abh}\Big\{\partial_\mu\big[(u_a A^\mu_b + A^\mu_a u_b)\Phi_h\big] - u_a\partial_\mu A^\mu_b\Phi_h - u_a A^\mu_b\partial_\mu\Phi_h\\
&- u_b\partial_\mu A^\mu_a\Phi_h - u_b A^\mu_a\partial_\mu\Phi_h + m_h A_{\mu a}A^\mu_b u_h\Big\} && (2.31)\\
d_Q T_1^3 =& + f^3_{abh}\Big\{(\partial_\mu A^\mu_a + m_a\Phi_a)u_b\Phi_h - m_h\tilde{u}_a u_b u_h\Big\} && (2.32)\\
d_Q T_1^4 =& - f^4_{hjk}\Big\{m_h u_h\Phi_j\Phi_k + m_j\Phi_h u_j\Phi_k + m_k\Phi_h\Phi_j u_k\Big\} && (2.33)\\
d_Q T_1^5 =& - f^5_{ahp}\Big\{\partial^\mu[u_a(\Phi_h\partial_\mu\varphi_p - \varphi_p\partial_\mu\Phi_h) - m_h A^\mu_a\varphi_p u_h]\\
&- (m_h^2 - \mu_p^2)u_a\Phi_h\varphi_p + 2m_h A^\mu_a u_h\partial_\mu\varphi_p + m_h\partial_\mu A^\mu_a u_h\varphi_p\Big\} && (2.34)\\
d_Q T_1^6 =& - f^6_{apq}\Big\{\partial^\mu[u_a(\varphi_p\partial_\mu\varphi_q - \varphi_q\partial_\mu\varphi_p)] + (\mu_q^2 - \mu_p^2)u_a\varphi_p\varphi_q\Big\} && (2.35)\\
d_Q T_1^7 =& - f^7_{abp}\Big\{\partial^\mu[(u_a A_{\mu b} + u_b A_{\mu a})\varphi_p]\\
&- (u_a\partial_\mu A^\mu_b + u_b\partial_\mu A^\mu_a)\varphi_p - (u_a A^\mu_b + u_b A^\mu_a)\partial_\mu\varphi_p\Big\} && (2.36)\\
d_Q T_1^8 =& + f^8_{abp}(\partial_\mu A^\mu_a + m_h\Phi_h)u_b\varphi_p && (2.37)\\
d_Q T_1^9 =& - f^9_{hjp}(m_h u_h\Phi_j + m_j u_j\Phi_h)\varphi_p && (2.38)\\
d_Q T_1^{10} =& - f^{10}_{hpq} m_h u_h\varphi_p\varphi_q && (2.39)\\
d_Q T_1^{11} =& 0. && (2.40)
\end{aligned}$$

The Q-vertex is given by the divergence terms above:

$$\begin{aligned}
T^\mu_{1/1} =& f_{abc}\Big[A_{\nu a}u_b(\partial^\nu A^\mu_c - \partial^\mu A^\nu_c) + \frac{1}{2}u_a u_b\partial^\mu\tilde{u}_c\Big] && (2.41)\\
&- f^1_{ahj}\Big[2u_a\Phi_h\partial^\mu\Phi_j + m_j A^\mu_a\Phi_h u_j - m_h A^\mu_a\Phi_j u_h\Big] && (2.42)\\
&- f^2_{abh}(u_a A^\mu_b + u_b A^\mu_a)\Phi_h && (2.43)
\end{aligned}$$

[9] Note the different sign for $d_Q T_1^5$ (2.34) compared to [2] & [16] and in (2.37) we have replaced Φ_a by Φ_h.

$$-\,f^5_{ahp}\Big[u_a(\Phi_h\partial^\mu\varphi_p-\varphi_p\partial^\mu\Phi_h)-m_hA^\mu_au_h\varphi_p\Big] \tag{2.44}$$

$$-\,2f^6_{apq}u_a\varphi_p\partial^\mu\varphi_q \tag{2.45}$$

$$-\,f^7_{abp}(u_aA^\mu_b+u_bA^\mu_a)\varphi_p. \tag{2.46}$$

The remaining terms must cancel out. Collecting terms proportional to $u_bA_{\mu a}A^\mu_c$, symmetric in a and c, we get the relation (no summation over c and a)

$$2m_bf^2_{acb}=(m_a^2-m_c^2)f_{abc}. \tag{2.47}$$

Hence, if $m_b=0$ and $f_{abc}\neq 0$ we must have

$$m_a=m_c. \tag{2.48}$$

For $m_b,m_h\neq 0$ we find

$$f^2_{abh}=\frac{m_b^2-m_a^2}{2m_h}f_{abh}. \tag{2.49}$$

Gathering terms of the form $A_{\mu a}u_h\partial^\mu\Phi_j$ we get

$$f^1_{ahj}=\frac{m_j^2+m_h^2-m_a^2}{4m_hm_j}f_{ahj} \tag{2.50}$$

and from $u_au_b\tilde{u}_c$ we obtain

$$m_b-f^3_{cab}-m_af^3_{cba}=m_c^2f_{abc}. \tag{2.51}$$

Taking use of all these results in the equation proportional to $\partial_\mu A^\mu_a\Phi_hu_j$ we arrive at

$$f^3_{ahj}=\frac{m_j^2-m_h^2+m_a^2}{2m_j}f_{ahj}, \tag{2.52}$$

and then from $u_h\Phi_j\Phi_k$ we obtain

$$f^4_{hjk}=0. \tag{2.53}$$

We have succeeded in expressing all couplings so far by f_{abc}. With these results all remaining terms, except for the Higgs couplings, cancel.

We next turn to the Higgs couplings. From $A^\mu_au_b\partial_\mu\varphi_p$ we find

$$f^7_{abp}=m_bf^5_{abp}=0 \text{ for } a>r \text{ or } b>r, \tag{2.54}$$

and from $\partial_\mu A^\mu_au_h\varphi_p$ we get

$$f^8_{abp}=-m_bf^5_{abp} \tag{2.55}$$

and zero for $b>r$. Finally the terms proportional to $u_a\Phi_h\varphi_p$ give

$$f^9_{ahp}=-\frac{\mu_p^2}{2m_a}f^5_{ahp},\ a\leq r \tag{2.56}$$

and zero for $a>r$. The terms proportional to $u_a\varphi_p\varphi_q$ lead to

$$f^{10}_{apq}=\frac{\mu_p^2-\mu_q^2}{m_a}f^6_{apq},\quad a\leq r \tag{2.57}$$

and zero for $a>r$. We see that the Higgs couplings are not completely fixed by first order gauge invariance. So far the Higgs couplings could even be set equal to zero, but then we would find a breakdown of gauge invariance at second order.

2.3. **Second order conditions.** We review in this subsection second order gauge conditions, since we refer to them in our following analysis again and again. For second order gauge invariance the gauge variation $d_Q T_2$ has to be looked for, where T_2 is derived from T_1 according to the causal construction, namely by splitting the causal distribution D_2. For finding $d_Q T_2$, we have first to calculate the causal distribution

$$D_2(x,y) = T_1(x)T_1(y) - T_1(y)T_1(x). \tag{2.58}$$

It has causal support (i.e. $\mathrm{supp} D_2(x,y) \subset \{(x-y)^2 \geq 0\}$) and must be decomposed into a retarded and advanced part: $D_2 = R_2 - A_2$, R_2 has support on the closed forward light cone, $\mathrm{supp} R_2 \subset \overline{V}^+$, A_2 on the backward light cone $\mathrm{supp} A_2 \subset \overline{V}^-$. In the end $T_2 = R_2 - R_2'$ can be calculated where $R_2'(x,y) := -T_1(x)T_1(y)$.

The distribution $D_2(x,y)$ is gauge invariant because

$$d_Q D_2(x,y) = [d_Q T_1(x), T_1(y)] + [T_1(x), d_Q T_1(y)] \tag{2.59}$$

$$= i\partial_\mu^x [T^\mu_{1/1}(x), T_1(y)] + i\partial_\mu^y [T_1(x), T^\mu_{1/1}(y)] \tag{2.60}$$

$$=: i\partial_\mu^x D^\mu_{2/1}(x,y) + i\partial_\mu^y D^\mu_{2/2}(x,y). \tag{2.61}$$

Since the retarded part R_2 agrees with D_2 on the forward light cone $x \in (V^+ \setminus \{0\})$ and similarly for $R^\mu_{2/1}, R^\mu_{2/2}$, gauge invariance of R_2 can only be violated by local terms proportional to $D^a \delta(x-y)$, that means local terms, containing $D^a \delta^4(x-y)$, with some partial differential operator D^a (see below), such terms can spoil gauge invariance and when they do so, we call them *anomalies*. They are due to the freedom in distribution splitting: If we have two splitting solutions r and $\tilde{r}$ for some numerical distribution d, then their difference

$$\tilde{r} - r = \sum_{|a|=0}^{\omega} \tilde{C}_a D^a \delta(x) \tag{2.62}$$

with

$$D^a = \frac{\partial^{a_1 + \ldots + a_m}}{\partial x_1^{a_1} \ldots \partial x_m^{a_m}} \quad \text{and} \quad |a| = a_1 + \ldots + a_m$$

must be a tempered distribution with point support at $x = 0$ ([2], (2.3.46)). For a general distribution its local part is not uniquely defined. But for tree graphs which we are going to consider, the local part is unique. The freedom in the splitting solution has to be fixed later in the theory by a physical normalisation[10] condition. If ω is strictly bigger than zero, we even have a sum of possible normalisation terms. This is the freedom in the splitting, causality determines the splitting solution only so far, but gauge invariance in many cases does the rest, providing the needed normalisation prescriptions. This means: in order to achieve second order gauge invariance we must in general cancel the anomalies or absorb them in normalisation terms if possible. These local normalisation terms $N_2, N^\mu_{2/1}, N^\mu_{2/2}$ must be chosen, if possible, such that

$$d_Q(R_2 + N_2) = \partial_\mu^x (R^\mu_{2/1} + N^\mu_{2/1}) + \partial_\mu^y (R^\mu_{2/2} + N^\mu_{2/2}) \tag{2.63}$$

holds. Then the theory is gauge invariant to second order. Note that the distribution $T_2 = R_2 + N_2 - R_2'$ then fulfills (2.63), too, because R_2' is clearly gauge invariant for the same reason as in (2.59) . The local terms on the right-hand side of (2.63),

[10] They are sometimes referred to as finite renormalisation terms.

which come from the causal splitting of the corresponding D-distributions, are the anomalies. To prove (2.63) we only have to consider its local part. Let R_2 be the splitting solution of D_2 obtained by replacing $D_m(x-y)$ by $D_m^{\rm ret}(x-y)$. Since d_Q operates only on the field operators, the local part on the left-hand side of (2.63) is only due to $d_Q N_2$. To calculate the anomalies on the right-hand side of (2.63) we start from

$$D^\mu_{2/1}(x,y) := [T^\mu_{1/1}(x), T_1(y)]. \tag{2.64}$$

The anomalies come from those terms in the Q-vertex $T^\mu_{1/1}$ that contain a derivative ∂^μ. These are the second and third term in (2.41), the first term in (2.42), the first two in (2.44) and the first in (2.45). Commuting the factors with derivative ∂^μ in these terms with all terms in $T_1(y)$ we get tree-graph contributions with four external legs (sectors). There is a second commutator to be considered, of course,

$$D^\mu_{2/2}(x,y) := [T_1(x), T^\mu_{1/1}(y)]. \tag{2.65}$$

giving together with (2.64) the gauge variation of D_2,

$$d_Q D_2(x,y) = \partial^x_\mu [T^\mu_{1/1}(x), T_1(y)] + i\partial^y_\mu [T_1(x), T^\mu_{1/1}(y)]. \tag{2.66}$$

In the following we will comment on some of the most important points in the concrete derivation of second order relations later referred to in the text. For a more detailed account we refer to the original paper [16] or [2].

2.3.1. *Sector $uA\tilde{u}u$: Jacobi identity.* In order to get the contributions for this sector we have to commute the first term in (2.41), [(2.41)/1] with the second one in (2.14), [(2.14)/2]:

$$\begin{aligned}[][(2.41)/1(x), (2.14)/2(y)] =& [-f_{abc} A_{\nu a} u_b \partial^\mu A^\nu_c(x), -i f_{def} A_{\lambda d}(y) u_b \partial^\mu \tilde{u}_c] \\ =& f_{abc} f_{cef} u_b(x) A_{\nu a}(x) \partial^\nu_x D(x-y) u_e(y) \partial^\mu_y \tilde{u}_f(y) \\ &+ f_{abc} f_{aef} u_b(x) D(x-y) \partial^\mu_x A^\nu_c(x) u_e(y) \partial^\nu_y \tilde{u}_f(y);\end{aligned} \tag{2.67}$$

the distributions appearing here can be trivially split, giving an expression containing

$$\partial^\nu_x D_{\rm ret}(x-y) \text{ and } D_{\rm ret}(x-y), \tag{2.68}$$

an expression that does, after application of the derivative ∂^μ_x in (2.66), not lead to anomalies. We will see that only terms containing a derivative ∂^μ lead to anomalies. Consider the contribution from the the second term in (2.41) and the second in (2.14):

$$\begin{aligned}[][(2.41)/2(x), (2.14)/2(y)] =& \, i f_{abc} f_{def} u_b(x) [A_{\nu a}(x) \partial^\mu_x A^\nu_c(x), A_{\lambda d}(y)] u_c(y) \partial^\nu_y \tilde{u}_f(y) \\ =& \, f_{abc} f_{cef} u_b(x) A_{\nu a}(x) \partial^\mu_x D(x-y) u_e(y) \partial^\nu_y \tilde{u}_f(y) \\ &- f_{abc} f_{aef} u_b(x) D(x-y) \partial^\mu_x A^\nu_c(x) u_e(y) \partial^\lambda_y \tilde{u}_f(y).\end{aligned} \tag{2.69}$$

After applying ∂^x_μ one gets a local term proportional to

$$\partial^\mu_x \partial^\mu_x D_{\rm ret}(x-y) = m^2 D_{\rm ret}(x-y) + \delta(x-y), \tag{2.70}$$

coming from the first summand in (2.69) and — as always — from interchanging x and y we get exactly the same contribution again. Thus we arrive at a first anomaly A_1

$$2 \cdot A_1(x,y) = -f_{abc} f_{cef} u_b A_{\nu a} u_e \partial_\nu \tilde{u}_f \delta(x-y). \tag{2.71}$$

and another anomaly A_2 arises from

$$
\begin{aligned}
[(2.41)/3(x),(2.14)/2(y)] =& -\frac{1}{2} i f_{abc} f_{def} u_a(x) u_b(x) A_{\nu d}(y) [\partial_x^\mu \tilde{u}(x), u_e(y)] \partial_y^\nu \tilde{u}_f(y) \\
=& \frac{1}{2} f_{abc} f_{cef} u_a(x) u_b(x) A_{\nu d}(x) \partial_x^\mu D(x-y) \partial_y^\nu \tilde{u}_f(y) \\
& - f_{abc} f_{aef} u_b(x) D(x-y) \partial_x^\mu A_c^\nu(x) u_e(y) \partial_y^\lambda \tilde{u}_f(y) \quad (2.72)
\end{aligned}
$$

and reads

$$
A_2(x,y) = f_{abc} f_{dcf} u_a u_b A_{\nu d} u_e \partial_\nu \tilde{u}_f \delta(x-y) \tag{2.73}
$$

which is already multiplied by two from interchange of x with y. Since there are no other terms with derivatives ∂^μ in this sector that could lead to further anomalies, we see that in order to preserve gauge invariance, anomaly A_1 has to cancel A_2. This will produce a restriction for the structure constants. Let us see which:

$$
2A_1 + 2A_2 = (-f_{abc} f_{cef} + f_{aec} f_{cbf} + f_{bec} f_{acf}) u_b u_e A_{\nu a} \partial^\nu \tilde{u}_f \delta(x-y) \stackrel{!}{=} 0. \tag{2.74}
$$

In the second term we have just interchanged the summation indices b and e. With antisymmetry we see that this condition is equivalent to the Jacobi identity

$$
f_{abc} f_{cef} + f_{eac} f_{cbf} + f_{bec} f_{caf} = 0. \tag{2.75}
$$

This is a very important result. Its says: reasonable physical gauge theories belong to real, reductive Lie algebras. A fact that in conventional formalism is assumed at the very beginning.

2.3.2. *Sector uAAA: A first normalisation term.* Here one gets

$$
\begin{aligned}
[(2.41)/2(x),(2.14)/1(y)] =& f_{abc} A_{\nu a}(x) u_b \big(f_{cef} A_{\alpha e}(x) \partial_x^\alpha A_f^\nu(y) \partial_y^\mu D(x-y) \\
& + f_{dcf} A_{\lambda d}(x) \partial_x^\nu A_f^\lambda(y) \partial_y^\mu D(x-y) \quad (2.76) \\
& - f_{cde} A_d^\nu(y) A_{\alpha e}(x) \partial_x^\mu \partial_y^\alpha D(x-y) \big). \quad (2.77)
\end{aligned}
$$

The last summand is of particular interest since the distribution D is of order $\omega = -2$ and $\partial^\mu \partial^\alpha D$ has singular order $\omega = 0$. As we know from the causal construction, whenever distributions have singular order ω there is freedom for normalisation in the splitting solution which can be seen from formula (2.62). That is: the retarded part $\partial^\mu \partial^\alpha D^{\mathrm{ret}} + \alpha_1 g^{\mu\alpha} \delta$ with a free constant α_1 contains a normalisation term as in (2.63). By compensation of this local term, α_1 is determined here [16], eq. (4.10) and we are left with

$$
N_1 = -\frac{i}{2} \beta_1 A_{\nu a} A_d^\nu A_{\alpha b} A_e^\alpha \delta(x-y) \tag{2.78}
$$

with

$$
\beta_1(a,b,d,e) = \frac{1}{2} f_{abc} f_{dec}. \tag{2.79}
$$

Such normalisation terms are quartic couplings required by gauge invariance.

2.3.3. *Sector $uA\Phi\varphi$: $f^6 = 0$.* Through compensation of local terms one shows $f^6 = 0$ and derives a second normalisation term

$$
N_2 = -\frac{i}{2} \beta_2 A_{\nu a} A_d^\nu A_{\nu d} \Phi_k \varphi_q \delta \tag{2.80}
$$

with

$$
\beta_2(a,d,k,q) = 2 f_{dac} f_{ckq}^5 - 8 f_{ajq}^5 f_{djk}^1. \tag{2.81}
$$

2.3.4. *Sector $uu\tilde{u}\varphi$: Diagonal Higgs-couplings $f^5 \neq 0$.* This is an important sector. We see here that f^5_{ajp} has to vanish for different a and j and one can derive here a relation that relates the Higgs-couplings for different gauge fields. It states that

$$f^5_{aap} = \frac{m_a}{m_b} f^5_{bbp} \tag{2.82}$$

and not less important

$$f^5_{aap} = 0 \quad \forall\, a > r. \tag{2.83}$$

2.3.5. *Sector $uA\Phi\Phi$: The first mass relation.* There is again a normalisation term

$$N_3 = -\frac{i}{2}\beta_3 A_{\nu a} A^\nu_d \Phi_j \Phi_h \delta \tag{2.84}$$

with

$$\beta_3(a,d,j,h) = 2 f_{dac} f^1_{chj} - 8 f^1_{ahk} f^1_{djk} - 2 f^5_{ahp} f^5_{djp}. \tag{2.85}$$

and a first condition relating Yang-Mills structure constants with corresponding gauge boson masses. To be clear, we always indicate the summations from now on:

$$\begin{aligned}\sum_{p=1}^{t} f^5_{ajp} f^5_{dhp} - f^5_{ahp} f^5_{djp} =& \sum_{c=1}^{r+s} \frac{m_j^2 + m_h^2 - m_c^2}{2 m_h m_j} f_{dac} f_{chj} \\ &- \sum_{k=1}^{r} \frac{m_k^2 + m_j^2 - m_a^2}{m_j m_k} \frac{m_k^2 + m_h^2 - m_d^2}{4 m_h m_k} f_{ajk} f_{dhk} \\ &+ \sum_{k=1}^{r} \frac{m_k^2 + m_h^2 - m_a^2}{m_h m_k} \frac{m_k^2 + m_j^2 - m_d^2}{4 m_j m_k} f_{ahk} f_{djk}, \quad (j \neq h).\end{aligned} \tag{2.86}$$

It will be crucial for our analysis that this relation (2.86) is symmetric under interchange of a, d and in h, j respectively. Interchanging two of these indices gives only a global sign change due to the antisymmetry of the structure constants: Indeed the first summand gets a sign change by interchanging d and a and/or interchanging h and j. The second summand is equal minus the third after this interchange, leading to a total sign change only. This comes also true in case f^5 is not zero, since also the left hand side in f^5 possesses the same symmetry.

We can also specify to special cases in the derivation. Then we go through the derivation from the beginning with this special choice of indices. Not all of them lead to different results. One special case with $a = j$ and $d = h$ ($j \neq h$) leads to

$$\begin{aligned}\sum_{p=1}^{t} f^5_{jjp} f^5_{jjp} =& \frac{1}{2m_h^2}\Bigg\{ \sum_{c=1}^{r+s} (m_j^2 + m_h^2 - m_c^2) f_{jhc} f_{jhc} \\ &- \sum_{k=1}^{r} \frac{m_k^4 - (m_j^2 - m_h^2)^2}{2 m_k^2} f_{jhk} f_{jhk} \Bigg\}.\end{aligned} \tag{2.87}$$

This relation shows, that f^5_{jjp} must be different from 0, because the r.h.s. is not zero. By means of relations (2.86) and (2.87) the Weinberg Salam model can be deduced [2], sect. 4.6, inserting two fields that must have the same mass (m_W), one with an other mass (m_Z) and a massless field (photon, $m_\gamma = 0$); we recommend the reader to study this case. But as one sees, the relations are far more general, giving constraints related to Yang-Mills structure constants on every physically possible gauge theory.

2.3.6. *Sector $uAA\Phi$.* Here one can find another relation that is more general than eq. (2.86) in that it allows to choose three indices to be massless or massive instead of two.

$$\begin{aligned} 2m_a \sum_{p=1}^{t} (f^5_{bhp} f^5_{cap} + f^5_{chp} f^5_{bap}) = & \sum_{d=1}^{r+s} \frac{2}{m_h} \Big[(m_d^2 - m_c^2) f_{bad} f_{chd} + (m_d^2 - m_b^2) f_{cad} f_{bhd} \Big] \\ & + \sum_{j=1}^{r} \Big[\frac{m_j^2 + m_h^2 - m_a^2}{m_h m_j^2} (m_c^2 - m_b^2) f_{ahj} f_{bcj} \\ & - \frac{m_j^2 + m_h^2 - m_b^2}{2 m_h m_j^2} (m_j^2 + m_a^2 - m_c^2) f_{bhj} f_{caj} \\ & - \frac{m_j^2 + m_h^2 - m_c^2}{2 m_h m_j^2} (m_j^2 + m_a^2 - m_b^2) f_{chj} f_{baj} \Big] \quad (b \neq c). \end{aligned} \tag{2.88}$$

The relation is only symmetric in the two indices c, b. Indices a and h are symmetric under interchange only

- if $m_c^2 = m_b^2$, then the second summand is zero.
- if the sum over the Higgs couplings f^5 vanish; this is particularly the case when a stands for a massless particle, $m_a = 0$.

The relation will therefore give most of the restrictions for a general gauge theory. In the derivation $b \neq c$ has been assumed. In this sector we will consider the following special cases from eq. (2.88):

Case $h = b \neq c$ yields

$$\begin{aligned} m_a m_b \sum_{p=1}^{t} f^5_{aap} f^5_{bbp} \delta_{ac} = & - m_c^2 \sum_{d>r} f_{abd} f_{bcd} + \sum_{j=1}^{r} \frac{f_{abj} f_{bcj}}{4 m_j^2} \Big[(m_j^2 - m_b^2) \times \\ & \times (3 m_j^2 - m_a^2 + m_b^2) - m_c^2 (m_j^2 + m_a^2 - m_b^2) \Big] \end{aligned} \tag{2.89}$$

In the special case $c = a$ we get

$$(m_a^2 - m_b^2) \sum_{d>r} (f_{abd})^2 = 0 \tag{2.90}$$

where (2.87) has been used. For $b \neq h \neq c$ we find a relation that is symmetric in b and c:

$$\begin{aligned} & \sum_{d>r} (m_c^2 f_{bad} f_{chd} + m_b^2 f_{cad} f_{bhd}) \\ = & \sum_{j=1}^{r} \frac{1}{4 m_j^2} \Big\{ f_{bhj} f_{caj} [(m_j^2 - m_b^2)(3 m_j^2 - m_a^2 + m_c^2) - m_h^2 (m_j^2 + m_a^2 - m_c^2)] \\ & + f_{baj} f_{chj} [(m_j^2 - m_c^2)(3 m_j^2 - m_a^2 + m_b^2) - m_h^2 (m_j^2 + m_a^2 - m_b^2)] \\ & - 2 f_{bcj} f_{ahj} (m_j^2 + m_h^2 - m_a^2)(m_b^2 - m_c^2) \Big\} \end{aligned} \tag{2.91}$$

where we corrected a misprint of the sign in the last column in previous publications [16], [2]. Therefore this relation is actually contained in eq. (2.88), yet it is sometimes helpful in its form and provides a consistency check later.

In the case $b = c$ we have to start the derivation from the beginning and can derive

$$\sum_{p=1}^{t}\Big[2m_a f^5_{bap}f^5_{bhp} - 2m_b f^5_{ahp}f^5_{bbp}\Big] = -\sum_{k=1}^{r}\frac{(m_k^2+m_a^2-m_b^2)(m_k^2+m_h^2-m_b^2)}{2m_h m_k^2}\times \\ \times f_{bak}f_{bhk} - 2\sum_{d=1}^{r+s}\frac{m_b^2-m_d^2}{m_h}f_{bad}f_{dbh}, \tag{2.92}$$

symmetric in h, a and b, d respectively. For $a = h \neq b = c$:

$$\sum_{p=1}^{t} m_b f^5_{aap}f^5_{bbp} = \frac{m_b^2}{m_a}\sum_{d>r}(f_{bad})^2 \\ + \sum_k \frac{(f_{bak})^2}{4m_a m_k^2}\Big[(m_k^2-m_b^2)(-3m_k^2-m_b^2+2m_a^2)+m_a^4\Big] \tag{2.93}$$

and finally for $h \neq a \neq b = c \neq h$

$$m_b^2\sum_{d>r}f_{bad}f_{bhd} = -\sum_{k=1}^{r}f_{bak}f_{bhk}\frac{1}{4m_k^2}\times \\ \times\Big[(m_k^2-m_b^2)(-3m_k^2+m_a^2-m_b^2+m_h^2)+m_a^2m_h^2\Big]. \tag{2.94}$$

2.3.7. *Sector $uA\varphi\varphi$.* Here we arrive at a new normalisation term

$$N_4 = -\frac{i}{2}\beta_4(a,b,p,q)A_{\nu a}A_b^{\nu}\varphi_p\varphi_q\delta \tag{2.95}$$

with

$$\beta_4(a,b,p,q) = -2f^5_{aap}f^5_{bbq}\delta_{ab} \tag{2.96}$$

which all remains true if $p = q$.

2.3.8. *Sector $uAA\varphi$.* One gets here another determination of $\beta_2(a,b,p,q)$. This is a consistency test with the result already obtained.

2.3.9. *Sector $u\Phi\Phi\varphi$.* This sector vanishes identically because $f^4 = f^6 = f^{10} = 0$.

2.3.10. *Sector $u\tilde{u}u\Phi$: The last mass relation.* One obtains here a new relation that looks very similar to eq. (2.86). But it is more general. It allows, as (2.91), to choose three indices to run through the full range $i = 1,\ldots,r+s$ and it can be shown that (2.86) is a special massive case of the relation found in this sector, an excellent consistency check again!

$$-\sum_{p=1}^{t}\Big[m_b f^5_{ajp}f^5_{dbp} + m_a f^5_{bjp}f^5_{dap}\Big] = \sum_{c=1}^{r+s}\frac{m_j^2-m_c^2+m_d^2}{2m_j}f_{abc}f_{dcj} \\ + \sum_{k=1}^{r}f_{ajk}f_{dbk}\frac{m_k^2+m_j^2-m_a^2}{4m_j m_k^2}(m_k^2-m_b^2+m_d^2) \\ - \sum_{k=1}^{r}f_{bjk}f_{dak}\frac{m_k^2+m_j^2-m_b^2}{4m_j m_k^2}(m_k^2-m_a^2+m_d^2). \tag{2.97}$$

The relation is symmetric in a, b and d, j; note a misprint in the sign in front of the Higgs couplings in previous publications [16], [2].

2.3.11. *Sector $u\varphi\varphi\varphi$.* This sector gives no further restriction. It vanishes identically.

2.3.12. *Sector $u\Phi\Phi\Phi$.* Here one brings out the normalisation term

$$N_5 = -\frac{i}{2}\beta_5(l,j)\Phi_l^2\Phi_j^2\delta \tag{2.98}$$

with

$$\beta_5(l,j) = -\sum_p \frac{\mu_p^2}{2}\Big(\frac{f^5_{aap}}{m_a}\Big)^2. \tag{2.99}$$

2.3.13. *Sector $u\Phi\varphi\varphi$: $\beta_6(h,p,q)$.* We are led to the normalisation constant N_6 and we find a very important and general relation for β_6 which will be needed for the Higgs-potential determined at third order.

$$N_6 = -\frac{i}{2}\beta_6(h,p,q)\Phi_h^2\varphi_p\varphi_q\delta \tag{2.100}$$

with

$$\beta_6(h,p,q) = (\mu_p^2+\mu_q^2)\frac{f^5_{aap}f^5_{aaq}}{m_a^2} - 6\sum_u \frac{f^5_{aau}}{m_a}f^{11}_{pqu}. \tag{2.101}$$

The pure Higgs-coupling f^{11} is up to now not specified. For special cases this can be achieved at third order [2], [25]. But one gets even more.

3. Third order: Higgs Potential

At third order gauge invariance we find a very general relation for the Higgs-potential for any gauge invariant theory, namely

$$\sum_{q=1}^{t} f^5_{aaq}\beta_6(j,p,q) = 2f^5_{aaq}\beta_5(a,j). \tag{3.1}$$

This can probably, together with (2.101), be linked to the Higgs-potentials normally set in in standard theories in an ad hoc manner. For the electroweak theory this has been managed ([2], sect. 4.5), setting t=1 which leads to

$$V(\varphi) = -ig^2\frac{\mu_p^2}{8m_a^2}(f^5_{aa1})^2\Big[\varphi^2+\sum_j\Phi_j^2+\frac{2m_a}{gf^5_{aa1}}\varphi\Big]^2. \tag{3.2}$$

This has the form of the conventional Higgs-potential.

4. A note on the electroweak theory

The group behind the electroweak theory is $SU(2)\times U(1)$. Its first factor $SU(2)$ has three generators, $U(1)$ is built up with one generator. Our Lie index therefore has values $a = 0,1,2,3$, corresponding to the hermitian fields

$$A_a^\mu : A^\mu, W_1^\mu, W_2^\mu, Z^0. \tag{4.1}$$

These fields are massive except for m_0, the photon mass. For each field a ghost field is introduced: u_0, u_1, u_2, u_3 and unphysical scalars Φ_1, Φ_2, Φ_3 and a physical scalar φ for which $d_Q\varphi = 0$ holds. By (4.1) a basis for the Lie algebra is selected, the mass eigenstate basis for asymptotic free states. The $\mathfrak{su}(2)$ (which is the corresponding Lie algebra for $SU(2)$) structure constants satisfy

$$f'_{a'b'c'} = \begin{cases} \epsilon_{a'b'c'} \text{ if } a'b'c' \in \{1,2,3\} \\ 0 \ \text{ otherwise.} \end{cases} \tag{4.2}$$

And we have an additional coupling f_{124} from the electroweak W's to the photon. We want to emphasise here that these structure constants, from the Lie structure alone, are only unique up to an orthogonal transformation

$$f_{abc} = O_{aa'} O_{bb'} O_{cc'} f'_{a'b'c'} \tag{4.3}$$

with a rotation matrix as e.g.

$$O_{aa'} = \begin{pmatrix} -\cos\theta & 0 & 0 & -\sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta & 0 & 0 & -\cos\theta \end{pmatrix}. \tag{4.4}$$

With θ we mean the Weinberg (electroweak mixing) angle. We have $f_{210} = \sin\theta$, $f_{321} = \cos\theta$ and $f_{310} = f_{320} = 0$ [2], [17] all others follow from antisymmetry.

5. Summary and outlook

Our structure constants are *real*, *totally antisymmetric* and fulfil the *Jacobi identity*. This means, the Yang-Mills parameters f_{abc} belong to a special class of Lie algebras: From the Cartan criteria we know that the adjoint representation of the algebra, defined by our f_{abc}'s, is semi-simple (for definitions see appendix A) . Equivalently the algebra is reductive, i.e. a direct sum of Abelian and compact simple Lie algebras.

With the link of the f_{abc}'s to structure constants we are directly able to meet with the adjoint representations and corresponding gauge potentials, discussed in the literature. Moreover, quantum gauge invariance is even more restrictive: from the relations (2.91), (2.97) and their special cases one specifies a physical basis for these Lie algebras. And some theories, even of Lie-type are ruled out, especially such with many massive gauge fields of different mass amount have a difficult stand in face of the number of mass relations they have to meet [26].

This means, we are equipped with a very strong tool for studying general massive gauge theories, as they are proposed in the literature. Our task is to adjust all the relations to the considered theory. For this we have to know its masses and the structure constants. One can of course try right away with concrete values for the f_{abc}'s, as we will do it in the case of $SU(5)$, but there is freedom for rotations (illustrated in the last section). Thus we will later look more generally at the structure constants in section 9.1, without assuming specific values. For the masses of a particular model we have to study the literature.

Part 2. **Massive and massless gauge fields**

6. Additional massless gauge fields

The relations of the electroweak theory are in the following considerations maintained; we want just add massless fields (gluons) to this theory and see what constraints there are. We take $r = 3$, as in the electroweak theory, but calculating now with open index s; f denotes — as above — the pure Yang-Mills structure constants, that are totally antisymmetric. Of course all of the (second) order gauge conditions have to be fulfilled. We first notice with antisymmetry, that diagonal elements have to be identically zero i.e. $f_{jjk} = f_{jja} = 0$; this holds also in the case $a = j$ and $k = j$. Now we consider the couplings of massless to massive gauge fields. Looking at (2.90) we see, that

$$f_{jkd} = 0, \quad \text{when } \; m_j \neq m_k \tag{6.1}$$

holds for a $d > r$, an index for a massless field. This excludes in the electroweak theory all couplings from massive to massless gauge fields except for f_{12d} (because $m_1 = m_{W^+} = m_{W^-} = m_2$). In our case, therefore, only the sum $\sum_{d>r}(f_{12d})^2$ can be different from zero. Condition (2.90) tells us further, that there is no coupling of one massive field to two massless ones. Just let only one index a or b be massive. Then we have, setting $a = j$ massive

$$m_j^2 \sum_{d>r}(f_{jbd})^2 = 0. \tag{6.2}$$

This is consistent with (2.86), (2.88) and (2.97); also from there we are led to this conclusion. We now proceed, taking the stated results into account, to (2.87), evaluating it for $h = 2$ and equate it with the expression for $h = 3$, which is possible because the left hand side is not dependent on h. Let $j = 1$; in the sum over k in (2.87) only non-diagonal elements count, we have

$$\begin{aligned}
&\frac{1}{2m_2^2}\Big\{\sum_{c=1}^{r+s}(m_1^2 + m_2^2 - m_c^2)f_{12c}^2 - \frac{m_3^4 - (m_1^2 - m_2^2)^2}{2m_3^2}f_{123}^2\Big\}\\
=&\frac{1}{2m_3^2}\Big\{\sum_{c=1}^{r+s}(m_1^2 + m_3^2 - m_c^2)f_{13c}^2 - \frac{m_2^2 - (m_1^2 - m_3^2)^2}{2m_2^2}f_{132}^2\Big\}.
\end{aligned} \tag{6.3}$$

Note, that the sum over c runs over massive *and* massless indices. By separation for the massive indices $1, 2, 3$ in the sums we get

$$\begin{aligned}
&\frac{1}{2m_2^2}(m_1^2 + m_2^2 - m_3^2)f_{123}^2 + \frac{1}{2m_2^2}\sum_{d>3} f_{12d}^2(m_1^2 + m_2^2) - \frac{m_3^4 - (m_1^2 - m_2^2)^2}{4m_3^2 m_2^2}f_{123}^2\\
&= \frac{1}{4m_2^2 m_3^2}\big\{2m_2^2(m_1^2 + m_3^2 - m_2^2) - m_2^2 + (m_1^2 - m_3^2)^2\big\}f_{123}^2
\end{aligned} \tag{6.4}$$

where we have used, that $f_{13d} = 0$. Collecting all factors of f_{123} this simplifies to

$$\frac{\sum_{d>3} f_{12d}^2}{f_{123}^2} = 2\frac{2m_2^2(m_1^2 - m_2^2) + m_3^2(m_3^2 - m_1^2)}{m_3^2(m_1^2 + m_2^2)}. \tag{6.5}$$

From this result we can see important implications. First, at least one f_{12c} must be different from zero, because the right hand side of (6.5) is in general not zero. Yet we know from first order gauge invariance (cf. (2.48)), that for structure constants with one mass zero ($m_c = 0$) and $f_{abc} \neq 0$, that the remaining masses m_a and m_b are of equal amount (degeneracy). Thus we can derive the relation

$$\frac{\sum_{d>3} f_{12d}^2}{f_{123}^2} = \frac{m_3^2}{m_1^2} - 1 \tag{6.6}$$

from (6.5). From (6.6) we further see, that only one of the f_{12c}'s can be different from zero leading to the electroweak theory which we want to maintain. Since there we have the relation [2], sect. 4.6:

$$\frac{f_{124}^2}{f_{123}^2} = \frac{m_3^2}{m_1^2} - 1, \tag{6.7}$$

we see that

$$f_{124}^2 = \sum_{d>3} f_{12d}^2 \tag{6.8}$$

or

$$\sum_{d>4} f_{12d}^2 = 0. \tag{6.9}$$

Note that the index d is not further specified by gauge invariance, which means, that in our gauge theory the photon is not signalised. We have to choose one index for the photon and do this in agreement with the old electroweak theory by setting it to four. Physically this means that the additional massless gauge fields (gluons) do not couple to the massive fields W and Z in a theory with 3 massive and n massless gauge fields.

Finally putting in (2.91) $a, b > r$; $c = 1$, $h = 2$ and $j = 1, \ldots, r$ we find

$$\sum_{d>3} f_{abd} f_{12d} = 0 \tag{6.10}$$

and see, because of $f_{12d} = 0$, $\forall d > 4$, that all f_{ab4} are identically zero, telling us, that the photon does not couple to the additional massless gauge bosons.

From second order gauge conditions these are the only restrictions we find. For the couplings of massless gauge bosons between themselves we find no restrictions. That means the gluons are neutral. The simple Standard Model with Lie algebra $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)$ is an example of such a theory. Let us now add massive gauge fields. One can proceed in a very similar manner adding only few new massive gauge bosons. Since for smaller theories this has already been worked out in [26], we jump directly to $\mathfrak{su}(5)$.

7. $\mathfrak{su}(5)$ — A REVIEW

Why do we still work with $\mathfrak{su}(5)$? Experimentally one already knows that $\mathfrak{su}(5)$ cannot be the ultimate unification algebra. For physical theories based on it predict a proton lifetime of [36][11]

$$\tau(p) \approx \frac{m_X^4}{g^2 m_p^5} = 10^{27} \div 10^{31} \ \text{ years,} \tag{7.1}$$

(m_X: unification mass scale, g: $\mathfrak{su}(5)$ coupling constant, m_p: proton mass) whereas experimental data (Particle Data Group, 1996) predicts for the dominant decay channel in the $\mathfrak{su}(5)$ model

$$\tau(p \longrightarrow \pi^0 e^+) \geq 10^{31} - 5 \times 10^{32} \ \text{ years.} \tag{7.2}$$

Nevertheless the $SU(5)$ group is believed to be the ingrediance of most grand unification groups (as $SO(10)$, E_6, E_8 and higher $SU(n)$ groups), since it is the smallest simple group that contains the Standard Model ([12], ch. 14) and has, as the only rank $n = 5$ candidate, except for $SU(3) \times SU(3)$, a complex representation which is necessary for the fermions. Therefore the $\mathfrak{su}(5)$ algebra should be compatible with our gauge conditions if such grand unification schemes make physical sense. On the other hand we are interested in the internal consistency of the $\mathfrak{su}(5)$ gauge theory, not in its relevance in phenomenology.

We start with an introduction to $\mathfrak{su}(5)$ (via [44]), herewith we are able to check our restrictions from gauge invariance. In the earliest model two representations of $\mathfrak{su}(5)$ are considered: the $\underline{24}$ representation for the super-strong breaking and the $\underline{5}$ representation. They yield a Higgs potential depending on the Higgs multiplets Φ (from $\underline{24}$) and H (from $\underline{5}$) and cross terms of these two fields [45]

$$\begin{aligned} V(H,\Phi) = &-\frac{1}{2}\mu^2 \operatorname{tr}\Phi^2 + \frac{1}{4}a(\operatorname{tr}\Phi^2)^2 + \frac{1}{2}b \operatorname{tr}\Phi^4 - \frac{1}{2}\nu^2 H^\dagger H + \frac{1}{4}\lambda(H^\dagger H)^2 \\ &+ \alpha H^\dagger H \operatorname{tr}\Phi^2 + \beta H^\dagger \Phi^2 H \end{aligned} \tag{7.3}$$

where the terms with α and β describe those cross terms; without them the minimisation is done in [42], giving rise to a minimal model. For mixed minimisation see [45], where the $\mathfrak{su}(5)$ Higgs potential is extensively studied. In the case $\alpha \neq 0$ and $\beta \neq 0$ the vacuum expectation values take on the form

$$\langle H \rangle = h \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}}, \end{pmatrix}, \ \langle \Phi \rangle = \nu \ \operatorname{diag}(1,1,1,-\frac{3}{2}-\frac{1}{2}\epsilon,-\frac{3}{2}+\frac{1}{2}\epsilon), \ \ \epsilon = \frac{3}{20}\frac{\beta}{b}\frac{h^2}{\nu^2} + \mathcal{O}(\frac{h^4}{\nu^4}). \tag{7.4}$$

The parameter ϵ is a combination of the parameters in (7.3) that comes out from minimisation of this Higgs potential. If ϵ is zero one speaks of *minimal* $\mathfrak{su}(5)$.

7.1. **Normalisation according to Georgi-Glashow.** We outline here the $\mathfrak{su}(5)$, as it was proposed by Georgi and Glashow in [31] and described at length by Langacker in his famous review article [36], sect. 3.3. which — in particular for this

[11]From the beginning though there have been attempts to prolong proton lifetime by enlarging $m_{X,Y}$ in order to save $\mathfrak{su}(5)$ [42] (note added in proof). At the beginning such was giving leeway to $\mathfrak{su}(5)$, but the better and better experimental value for $\sin^2\theta_W(m_X)$ was restraining it soon. Therefore despite all those modifications in the literature it is today widely believed that $\mathfrak{su}(5)$ is physically definitely ruled out. Mathematically this is not so and explains why many extensions of the model are still discussed [41], [37].

section — originated in [42]. The gauge fields are written as a sum over the generalised Gell-Mann matrices λ^i as they are listed in [29] in the case of $\mathfrak{su}(5)$ and one gets

$$A = \sum_{i=1}^{24} \frac{A^i \lambda^i}{\sqrt{2}} = \begin{pmatrix} G^1_1 - \frac{2B}{\sqrt{30}} & G^1_2 & G^1_3 & \overline{X}^1 & \overline{Y}^1 \\ G^2_1 & G^2_2 - \frac{2B}{\sqrt{30}} & G^2_3 & \overline{X}^2 & \overline{Y}^2 \\ G^3_1 & G^3_2 & G^3_3 - \frac{2B}{\sqrt{30}} & \overline{X}^3 & \overline{Y}^3 \\ X_1 & X_2 & X_3 & \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & -\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{pmatrix}, \tag{7.5}$$

$$\operatorname{tr} A = 0 \tag{7.6}$$

with

$$\begin{aligned} G^1_1 &= G^3/\sqrt{2} + G^8/\sqrt{6} \\ G^2_2 &= -G^3/\sqrt{2} + G^8/\sqrt{6} \\ G^3_3 &= -2G^8/\sqrt{6}. \end{aligned} \tag{7.7}$$

The normalisation depends on the representation one works with: In the trace condition [5], (21.5.1)

$$\operatorname{tr}\{T_\alpha T_\beta\} = N_D \delta_{\alpha\beta} \tag{7.8}$$

N_D can take any value. For the fundamental representation its value in $\mathfrak{su}(n)$ is $N_D = 2$ [24]. The factor

$$\frac{1}{\sqrt{2n(n+1)}} = \frac{1}{\sqrt{2}\sqrt{30}} \tag{7.9}$$

is a special normalisation due to [10], eq. (13.4). The normalisation in the literature defers sometimes from this specific one.[12] Only the trace condition is demanded, since the basic fields are hermitian. The off-diagonal G's are combined like in the $\mathfrak{su}(3)$ case as

$$\begin{aligned} G^1_2 &= G^1/\sqrt{2} + iG^2/\sqrt{2} \\ G^2_1 &= G^1/\sqrt{2} - iG^2/\sqrt{2} \end{aligned} \tag{7.10}$$

and for the electroweak part

$$W^\pm = (W^1 \pm iW^2)/\sqrt{2} \tag{7.11}$$

and the same is done with the super-heavy particles X and Y in this model

$$\overline{X}^1 = (X^1 + iX^2)/\sqrt{2},\ X^1 = (X^1 - iX^2)/\sqrt{2}\ \text{etc.} \tag{7.12}$$

For the off-diagonal entries in A (7.5) there is no freedom for rotation; they correspond directly to the hermitian generators for the gauge bosons. Those generators can be rotated however if they are part of the same multiplet with mass degeneracy. This freedom allows for the rotation (7.11) and (7.12). The diagonal generators can be rotated, as this is done in the electroweak case in order to obtain the neutral Z

[12] Cf. [8], [7], [36] (for physical normalisations), [36] and [10] (for general $\mathfrak{su}(n)$ traceless generators). Especially the B and W^3 fields are differently normalised.

boson and the photon A^μ ([7], (6.25)) from the diagonal W^3 and B generators with the Weinberg angle θ_W

$$\begin{aligned} Z_\mu &= \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \\ A_\mu &= \sin\theta_W W^3_\mu + \cos\theta_W B_\mu. \end{aligned} \tag{7.13}$$

In $\mathfrak{su}(5)$ we have four diagonal generators: two correspond to the diagonal gluon generators and the other two look like

$$\begin{aligned} W_3 &= \mathrm{diag}(0,0,0,1/\sqrt{2},-1/\sqrt{2}) \\ B &= \mathrm{diag}\frac{1}{\sqrt{30}}(2,2,2,-3,-3). \end{aligned} \tag{7.14}$$

We denote them here as in the electroweak case, because the same rotation (7.13) is carried out leading to the generators for the Z and the photon A^μ (we denote it as γ for short)

$$Z = \begin{pmatrix} +\frac{2}{\sqrt{30}}\cos\theta_W & & & & \\ & +\frac{2}{\sqrt{30}}\cos\theta_W & & 0 & \\ & & +\frac{2}{\sqrt{30}}\cos\theta_W & & \\ & 0 & & -\frac{1}{\sqrt{2}}\sin\theta_W - \frac{3}{\sqrt{30}}\cos\theta_W & \\ & & & & \frac{1}{\sqrt{2}}\sin\theta_W - \frac{3}{\sqrt{30}}\cos\theta_W \end{pmatrix}, \tag{7.15}$$

$$\gamma = \begin{pmatrix} -\frac{2}{\sqrt{30}}\sin\theta_W & & & & \\ & -\frac{2}{\sqrt{30}}\sin\theta_W & & 0 & \\ & & -\frac{2}{\sqrt{30}}\sin\theta_W & & \\ & 0 & & \frac{1}{\sqrt{2}}\cos\theta_W + \frac{3}{\sqrt{30}}\sin\theta_W & \\ & & & & -\frac{1}{\sqrt{2}}\cos\theta_W + \frac{3}{\sqrt{30}}\sin\theta_W \end{pmatrix}. \tag{7.16}$$

Based on these generators one can calculate the masses for the gauge bosons.

7.2. **Mass relations in $\mathfrak{su}(5)$.** It is possible to express the gauge boson masses in terms of the parameters of the Higgs potential and the coupling constants therein. For the mass relations we first consider the case when in (7.4) $\epsilon = 0$ (minimal $\mathfrak{su}(5)$) and $h = 0$ for simplification. For getting the mass terms one starts with the expression for the covariant derivative for Φ which is in the adjoint representation [13], eq. (14.30)

$$\begin{aligned} D_\mu\Phi &= \partial_\mu\Phi + ig[A_\mu,\Phi] \\ &= D_\mu\Phi' + ig[A_\mu,\langle\Phi\rangle]. \end{aligned} \tag{7.17}$$

A_μ denotes here the matrix (7.5) and $\Phi' = \Phi - \langle\Phi\rangle$ is the shifted Higgs field in matrix form [13], eq. (14.29). As in the Standard Model one interprets $|D_\mu\Phi|^2$ as a term for kinetic energy. Therein the scalar $g^2|[A_\mu,\langle\Phi\rangle]|^2$ is the desired mass term. Looking for expressions for the masses of the super-heavy particles X and Y we have to project out the relevant contributions from the A-matrix (7.5). For the first

X-particle these are the generators

$$\lambda_a = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \lambda_b = \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.18}$$

We then calculate the commutators[13]

$$[\lambda_a, \langle\Phi\rangle] = -[\lambda_b, \langle\Phi\rangle] = \begin{pmatrix} 0 & 0 & 0 & -2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.19}$$

Taking the trace of the square, this gives the value for the mass of the X-bosons

$$m_X^2 = \frac{25}{8} g^2 \nu^2. \tag{7.20}$$

One gets the same value for the mass of the Y-bosons, when $h = 0$, as it was assumed so far. In the more general case with two Higgs-multiplets we have the mass formula [42], [5], eq. (21.1.17) [14]

$$\mathcal{M}_{ab} = g^2[-\frac{1}{8} \operatorname{tr}\{[\lambda^a, \langle\Phi\rangle][\lambda^b, \langle\Phi\rangle]\} + \frac{1}{8} \langle H\rangle^\dagger \{\lambda_a, \lambda_b\} \langle H\rangle]; \tag{7.21}$$

for m_X this does so far not change anything, but to m_Y will be added a term from H. If we also take cross terms of the two Higgs multiplets into account [30], i.e. $\epsilon \neq 0$ in (7.4) we get for both masses an ϵ correction. Finally we get the mass relations, as one can find them in the literature [12], [44] up to normalisation:

(1) $m_W^2 = \frac{1}{8} g^2 h^2 + \frac{1}{4} \epsilon^2 g^2 \nu^2$
(2) $m_Z^2 = \frac{8}{15} g^2 h^2$
(3) $m_X^2 = \frac{1}{8}(5 + \epsilon)^2 g^2 \nu^2$
(4) $m_Y^2 = \frac{1}{8}(5 - \epsilon)^2 g^2 \nu^2 + \frac{1}{4} g^2 h^2$
(5) eight massless gluons
(6) one massless photon.

7.3. **Charges of the $\mathfrak{su}(5)$ bosons.** Apart from the mass relations for the gauge bosons we will need information about the structure constants. One can take the program in appendix B to obtain structure constants for the normalisation we work with. Yet there is freedom and one is not sure whether exactly these values for the structure constants come out from gauge invariance. In order to circumvent this problem, we will adopt a more general treatment for the structure constants in the following, based on charge conservation and gauge invariance itself. Therefore we have a glance at charges in $\mathfrak{su}(5)$ in this section. As for smaller $\mathfrak{su}(n)$ algebras one

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[13] A faster way ([13], Problems and Solutions) is to write the commutators like

$$[A_\mu, \langle\Phi\rangle]^j_k = (A_\mu)^j_k(\Phi_k - \Phi_j)$$

where

$$\langle\Phi\rangle^j_k = \Phi_k \delta^j_k.$$

Here one sees directly that if $\Phi_k = \Phi_j$ then the gauge field is massless.

[14] A mass formula for generators in the Cartan-Weyl basis (cf. appendix A) can be found in [43], eq. (9.1).

can calculate the charge operator Q as a superposition of the diagonal generators of the algebra. In the fundamental representation Q looks like [13], (14.12)

$$Q = \mathrm{diag}(-1/3, -1/3, -1/3, 1, 0). \tag{7.22}$$

The charges of the particles in the A-matrix A_{ij} are $Q(A_{ij}) = Q_i - Q_j$ where Q_i, Q_j are the diagonal entries in Q. Gluons and W's remain neutral/charged as they are in the Standard Model, the new X and Y particles carry electric charges:

$$Q_X = -1/3 - 1 = -4/3, \quad Q_Y = -1/3 - 0 = -1/3. \tag{7.23}$$

The charges are different from ± 1 and each multiplet has different values. This is helpful when considering the coupling structure for the model. With charge conservation one can look for possible couplings between the gauge bosons which will reduce the number of parameters in our second order relations considerably. Before looking at the coupling structure, let us agree upon a little remake in the notation.

7.4. **Notation.** Instead of $\overline{X}^1, \overline{X}^2, \overline{X}^3$ and X^1, X^2, X^3 we denote the X particle multiplet just with the numbers $i = 1, \ldots, 6$ and similarly the Y particles with $j = 1, \ldots, 6$ if only one of the two multiplets is involved. In case both are involved we denote them by $X^i,\ i = 1, \ldots, 6$ and $Y^j,\ j = 1, \ldots, 6$ respectively. This notation is more general; the notation in (7.5) could be misleading.

8. A direct check with structure constants

In appendix A.6 we have derived totally antisymmetric structure constants for $\mathfrak{su}(5)$, starting with a Cartan-Weyl basis in order to make the link to the literature [29]. We will now assign to each of the indices a generator with a corresponding mass and look whether this agrees with gauge invariance. For this assignment we have not a symmetrical coupling of the photon and the Z, since our Z couples here only to the Y particles not to the X's! And there is freedom in the Cartan-Weyl subalgebra to superpose its generators. Nevertheless it is worth checking these assignments, since they are tempting by physical arguments; later we take advantage of the freedom for rotation among gauge bosons of the same mass multiplet or by rotating the diagonal elements as in (7.14).

8.1. **Check without and with rotations.** Since $\mathfrak{su}(3)$ is a subalgebra of $\mathfrak{su}(5)$, the structure constants of $\mathfrak{su}(3)$ must of course be contained in our list. And indeed

$$f_{123}, f_{147}, f_{156}, f_{246}, f_{257}, f_{345}, f_{367}, f_{458}, f_{678}$$

agree with the nine non-vanishing structure constants of $\mathfrak{su}(3)$, that are normally derived by commuting the eight Gell-Mann matrices λ_i, see appendix A.7. Therefore we assign the first eight numbers (generators) to the λ_i, i.e. indices $1, \ldots, 8$ will denote the gluons in this check. For the generators of the electroweak theory we can derive the strength of the coupling when we know the value for the Weinberg angle and this we know for $\mathfrak{su}(5)$

$$\sin^2 \theta_W = 3/8 \tag{8.1}$$

without including (re-)normalisation effects [31], a result that is exclusively based on group theory and already contained in the values of the structure constants. The electroweak couplings correspond to our $f_{(15)(22)(23)}$ and $f_{(22)(23)(24)}$. In the electroweak case [2], sect. 4.6 one sees directly the Z and the A^μ without the rotation (7.14). It seems plausible therefore assigning the two remaining diagonal generators directly to the Z and the A^μ in a first check. By assigning the index 22 to the W^+

and W^- to 23, we are thus inclined, associating the photon to the generator with index 15 and to 24 the Z^0 or vice versa.

There are now only the generators with indices $9, \ldots, 14$ and $16, \ldots, 21$ not assigned. We see, that we have assigned the gluons in such a way that they either couple to the class of generators $9, \ldots, 14$ or to $16, \ldots, 21$; they never couple to two generators from both classes. That's why we will assign the first 6 generators $9, \ldots, 14$ to the X bosons and $16, \ldots, 21$ to the Y particles. For their masses we do not insert the values given above, just the mass degeneracies count. The six X particles have the same mass m_X and the other six super-heavy fields Y the common mass m_Y.

Now we go to (2.97) and adjust it in the following way. We choose the case where we insert only X-particles for the free indices: $a = 16$, $b = 17$, $d = 18$ and $j = 19$ and divide the relation by $m_{16} = m_X$. In our choice $a \neq d \neq b$ holds, hence the left hand side is zero

$$\begin{aligned} 0 = & - \frac{2m_X^2}{2m_X^2} \Big[\sum_{c>r} f_{(16)(17)c} f_{(18)(19)c} + f_{(16)(17)Z} f_{(18)(19)Z} \Big] + \frac{m_Z^2}{2m_X^2} f_{(16)(17)Z} f_{(18)(19)Z} \\ & + f_{(16)(19)Z} f_{(18)(17)Z} \frac{m_Z^2}{4m_X^2} - f_{(17)(19)Z} f_{(18)(16)Z} \frac{m_Z^2}{4m_X^2}. \end{aligned} \tag{8.2}$$

The three non-vanishing couplings to the Z are

$$f_{(16)(17)Z},\ f_{(18)(19)Z} \text{ and } f_{(20)(21)Z} = +\frac{5}{2\sqrt{10}}$$

and the sum of the couplings to the massless gauge bosons, the gluons and the photon, we know as well:

$$\begin{aligned} \sum_{c>r} f_{(16)(17)c} f_{(18)c(19)} = & - f_{3(16)(17)} f_{3(18)(19)} - f_{8(16)(17)} f_{8(18)(19)} \\ & - f_{(15)(16)(17)} f_{(15)(18)(19)} \\ = & + \frac{1}{4} - \frac{1}{12} - \frac{1}{24} \\ = & + \frac{1}{8}. \end{aligned} \tag{8.3}$$

Equation (8.2) leads then to

$$\frac{m_Z^2}{m_X^2} = +\frac{8}{5} \tag{8.4}$$

which is surely not an expected relation for a grand unified model.

Next is to take advantage of the freedom for rotation for the diagonal generators (7.14); the structure constants after a special rotation with $\sin^2\theta_W = 3/8$ can be found in appendix A.8. Also with this set of structure constants we find contradictions. In addition to rotations among the diagonal generators we have freedom for rotations among the generators of the same mass multiplet. We can also not be sure to see the “right” Weinberg angle. Therefore, it is better to proceed without explicit values for the structure constants which we do in the following.

9. Coupling structure in Georgi-Glashow $\mathfrak{su}(5)$

9.1. **Possible couplings by virtue of charge conservation.** We have studied in the last sections the Georgi-Glashow model in order to be sure what masses and charges the bosons in this special $\mathfrak{su}(5)$ scheme have. We will here not work with specific values for the structure constants as one could calculate them easily by commuting the generalised Gell-Mann matrices. When doing this one finds that our second order relations cannot be fulfilled. Yet, as noticed, the freedom for the diagonal generators could be a stumbling block and one cannot be conclusive at all: We would have to allow for rotations which rakes us even more unknown parameters! It is therefore advisable to let the structure constants be as general as possible. Here they ought to be antisymmetric and real only. The Jacobi identity is not used explicitly.

In order to discuss charge conservation we suppose the coupling structure of the charged super-heavy fields as $X^+ = X^1 + iX^2$ and $X^- = X^1 - iX^2$, $W^\pm = W^1 \pm iW^2$ which is mirrored by A (7.5) in the Georgi-Glashow model. From charge conversation we know which couplings among them are possible. What can we say then on the coupling structure among the hermitian fields as W^1, W^2, X^1, X^2 etc.?

From charge conservation we know e.g. that $f_{X^+X^-W^\pm} = 0$; this implies $f_{X^1X^2W^2} = 0$ and $f_{X^1X^2W^1} = 0$, because

$$\begin{aligned} f_{X^+X^-W^+} =& f_{X^1X^1W^1} + if_{X^1X^1W^2} + if_{X^2X^1W^1} - f_{X^2X^1W^2} \\ & - if_{X^1X^2W^1} + f_{X^1X^2W^2} - f_{X^2X^2W^1} + if_{X^2X^2W^2} \\ =& -2if_{X^1X^2W^1} + 2f_{X^1X^2W^2} \end{aligned} \tag{9.1}$$

due to antisymmetry of the f's. This shows that $W^{1,2}$ cannot couple to two X bosons if they are charged in the way it was described in subsection 7.3. The same holds for the couplings of the form $f_{X^+X^-Y^\pm}$. Hence $f_{X^1X^2Y^i} = 0$, $i = 1, \dots, 6$. The only left massive boson is then the Z^0. Let us see whether this can couple to two X. We assume $f_{X^+X^-Z^0} \neq 0$ and then

$$\begin{aligned} f_{X^+X^-Z} &= f_{X^1X^1Z} - if_{X^1X^2Z} + if_{X^2X^1Z} + f_{X^2X^2Z} \\ &= -2if_{X^1X^2Z}. \end{aligned} \tag{9.2}$$

So if as assumed $f_{X^+X^-Z}$ is not zero we have also $f_{X^1X^2Z^0} \neq 0$. The arguments above hold in the case of two Y's as well of course, only the amount of charge is different. From charge conservation, couplings of the form $f_{X^+Y^-W^\pm}$ are possible, f_{XYZ} not. But here the couplings for the hermitian generators cannot be coercively ruled out. We will show this using gauge invariance. Similar arguments though can be used for the couplings of the form f_{XXX} or f_{YYY}.

9.2. **Additional constraints from gauge invariance.** In this subsection we will show how far gauge invariance can restrict our coupling structure in $\mathfrak{su}(5)$. We start with couplings of the form f_{XWZ}. We go back to (2.89) and analyse it for $a = X^2$, $b = W^1$ and $c = W^2$. When the masses m_X and m_W are different we have no couplings f_{XWd}, with $d > r$ due to eq. (2.90):

$$\begin{aligned} 0 =& + \sum_j f_{X^1W^1j} f_{W^1W^2j} [(m_j^2 - m_W^2)(3m_j^2 - m_X^2 + m_W^2) \\ & - m_W^2(m_j^2 + m_X^2 - m_W^2)] \\ =& f_{X^1W^1Z} f_{W^1W^2Z} [3m_Z^4 - m_Z^2 m_X^2 - 3m_Z^2 m_W^2] \end{aligned} \tag{9.3}$$

where the second equality holds because we know from the electroweak relation (cf. below 9.3), that W^1 and W^2 couple only to the Z. Similar arguments hold for f_{YWZ}, f_{XWW}, f_{XZZ} and the like. We conclude the discussion of purely massive couplings and show that $f_{XYZ} = 0$. We go back to (2.89) and analyse it for $a = X^2$, $b = X^1$ and $c = Y^1$:

$$\begin{aligned}0 =&(3m_W^2 - 3m_X^2 - m_Y^2)[f_{X^1X^2W^1}f_{X^1Y^1W^1} + f_{X^1X^2W^2}f_{X^1Y^1W^2}]\\&+ (3m_Z - 3m_X^2 - m_Y^2)f_{X^1X^2Z}f_{X^1Y^1Z}\\&- m_Y^2\sum_i f_{X^1X^2X^i}f_{X^1Y^1X^i}\\&+ (2m_Y^2 - 3m_X^2)\sum_j f_{X^1X^2Y^j}f_{X^1Y^1Y^j}\end{aligned} \tag{9.4}$$

where we did not write couplings to massless indices here because of (2.90) again. One can repeat the same choice for all combinations of the particles of the two multiplets being left with a set of equations that restricts the (sums of) structure constants. But this is not conclusive, since it could be possible that the structure constants adopt any value, not just in the range of ± 2 because in a relation as e.g. in the electroweak case

$$\Big(\frac{f_{W^1W^2\gamma}}{f_{W^1W^2Z}}\Big)^2 = \tan^2\theta \tag{9.5}$$

the mixing angle is not necessarily given by the value $\sin^2\theta = 3/8$, but can even lie close to $\pi/2$; $\tan^2\theta$ can then be a huge number. This means that the structure constants could be of values near to the mass ratios m_X/m_Z, m_Y/m_Z etc. Later we will analyse such systems of equations in detail: One can check any mass degeneracy and is not forced to use arguments based on charge conservation as one does not know in general how these new gauge particles are charged. For the Georgi-Glashow model though we can take the results from section 9.1 into account in eq. (9.1).

and are left with the relation

$$0 = f_{X^1X^2Z}f_{X^1Y^1Z}(3m_Z - 3m_X^2 - m_Y^2) \tag{9.6}$$

since in the other products of structure constants always one factor is zero. We end up with a mass relation that is not true for the Georgi-Glashow model and conclude that $f_{X^1Y^1Z}$ is zero, since $f_{X^1X^2Z}$ is not.

So far we have considered purely massive couplings. For couplings between massive and massless bosons gauge invariance alone does constrain the coupling structure. In this case we do not need charge conservation. For example going to eq. ([2], (4.4.43)), setting $a = h = X^1$ (index 1) and $b = \gamma$ we note that the left hand side is zero because $f^5_{\gamma\gamma p}$ is zero, and we see that

$$\begin{aligned}0 =& -\sum_{d=1}^r (f_{1\gamma d})^2\frac{(m_d^2 + m_X^2)^2}{2m_X^2m_d^2} + 2\sum_{d=1}^r (f_{\gamma 1d})^2\frac{m_d^2}{m_X^2}\\=&\sum_{d=1}^r (f_{1\gamma d})^2\Big[2\frac{m_d^2}{m_X^2} - \frac{(m_d^2 + m_X^2)^2}{2m_X^2m_d^2}\Big]\end{aligned} \tag{9.7}$$

if we now allow the massive index d to run through X^i, $i = 1, \dots, 6$ only, this relation is best fulfilled:

$$0 = \Big[2\frac{m_X^2}{m_X^2} - \frac{4m_X^4}{2m_X^4}\Big] \sum_{i=1}^{6} (f_{1\gamma X^i})^2. \tag{9.8}$$

If the sum over d ran over other massive indices ($d = Z^0$, $W^{1,2}$, Y^j, $j = 1, \dots, 6$) beside the X bosons, the mass factors would not compensate. Thus these structure constants vanish. The same holds true if we replace the photon index γ with a gluon index here, i.e. couplings that are forbidden by "colour conservation" can also be ruled out by gauge invariance: we show here that couplings of the form $f_{\lambda\lambda Z}$ can be suspended. This can best be demonstrated using eq. (2.89) with the content $a = c = \lambda_1$, $b = \lambda_2$, which leads to

$$0 = 0 - \frac{3}{4} f^2_{\lambda^1\lambda^2 Z} m_Z^2. \tag{9.9}$$

Since m_Z is not zero and because we can choose here any two gluon indices we have reached our claim, i.e. $f_{\lambda\lambda Z} = 0$. Also the type $f_{WW\lambda}$ is easily shown to vanish: in (2.86) we put $d = \lambda^1$, $a = \lambda^2$, $h = W^1$ and $j = W^2$ and see

$$0 = \sum_{i=1}^{8} f_{\lambda^1\lambda^2\lambda^i} f_{\lambda^i W^1 W^2} \cdot 1 - 0 + 0. \tag{9.10}$$

And because the gluons do self-couple and since we can here go through all gluon indices we have necessarily $f_{\lambda WW} = 0$. As seen in (2.90) all couplings for two massless to one massive gauge boson do vanish generally. The same is true for the coupling of one massless boson to two bosons of different non-vanishing masses, as for $f_{\lambda XY}$. We summarise our possible non-vanishing couplings in table 1. In the sequence of this work we will look for relations among these couplings and the masses involved. We start with the electroweak relation and look whether it still holds with this coupling structure beyond the Standard Model.

Couplings of massless gauge bosons	Purely massive couplings
$f_{\lambda\lambda\lambda}$	f_{WWZ}
$f_{YY\lambda}$	f_{XXZ}
$f_{XX\lambda}$	f_{YYZ}
$f_{YY\gamma}$	f_{XYW}
$f_{XX\gamma}$	
$f_{WW\gamma}$	

Table 1. Possible non-vanishing structure constants. The same index always means a different particle of the same multiplet, e.g. WW corresponds to W^1W^2 and $\lambda\lambda\lambda$ means $\lambda_i\lambda_j\lambda_k$ for $i < j < k \in \{1, \dots, 8\}$.

9.3. **Electroweak relation.** We show here that with the couplings of table 1 the relation (6.6) still holds true, also if for the Weinberg angle a different value can result. In this subsection we briefly dwell on this point and show that the electroweak relation indeed formally remains true, underlining that the couplings make sense. In 9.2 we did already use the fact that W^1 and W^2 couple only to the Z. One can see here the consequences if we had not assumed this. The electroweak relation would be changed and we would not see the subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ anymore. Actually we should see again that

$$\frac{f^2_{W^1W^2Z}}{f^2_{W^1W^2\gamma}} = \frac{m_Z^2}{m_W^2} - 1 \tag{9.11}$$

holds. For this we go to (2.87) with $j = W^1$, $h = W^2$ and equate it with the same equation but setting $h = Z$, which is always possible with the same index j here,

$$\begin{aligned} &\frac{m^2_{W_1} + m^2_{W_2}}{2m^2_{W_2}} \left(\sum_{c>r} f^2_{W^1W^2c} + f^2_{W^1W^2Z} \right) \\ &- \frac{m_Z^2}{2m_W^2} f_{W^1W^2Z} - \frac{m_Z^4 - m^4_{W_1} + m^4_{W_2} - 2m^2_{W_1}m^2_{W_2}}{4m_Z^2} f_{W^1W^2Z} \\ &\stackrel{!}{=} \frac{1}{2} f^2_{W^1ZW^2} - \frac{1}{2m_Z^2} \frac{m^4_{W_2} - m^4_{W_1} + m_Z^4 - 2m^2_{W_1}m_Z^2}{2m_W^2} f_{W^1ZW^2}. \end{aligned} \tag{9.12}$$

and we arrive again at

$$\frac{\sum_{c>r} f^2_{W^1W^2c}}{f^2_{W^1W^2Z}} = \frac{4m^2_{W_2}(m^2_{W_1} - m^2_{W_2}) + 2m_Z^2(m_Z^2 - m^2_{W_2})}{m_Z^2(m^2_{W_1} + m^2_{W_2})}; \tag{9.13}$$

we can here put $m_{W_1} = m_{W_2}$ (cf. [2], (4.3.19)). We see that only electroweak particles are involved here and find again the electroweak relation, as claimed, making apparent, that we have indeed included $\mathfrak{su}(2) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(5)$. And as there are no restrictions for the gluons ($\mathfrak{su}(3)$) from gauge invariance: we have even the inclusion $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(5)$. This shows that our coupling structure makes sense in the electroweak case. Had we allowed couplings of the form $f_{W^1W^2X}$ or $f_{W^1W^2Y}$ we would have a much more complicated relation than (9.11).

10. Explicit check of $\mathfrak{su}(5)$

We are ready now for the examination of the Georgi-Glashow model in the most general setting. Let us summarise the field content we work with below

(1) 8 massless gluons
(2) 1 massless photon
(3) 2 massive charged W bosons with mass m_W
(4) 1 massive uncharged Z^0 with mass m_Z
(5) 6 massive charged X bosons with masses m_X
(6) 6 massive charged Y bosons with masses m_Y.

For the structure constants we look at table 1. All of the second order relations have to be fulfilled simultaneously. We start with the purely massive couplings, since gauge invariance then is very restrictive. This is especially true when one inserts couplings between gauge bosons with four different masses. That means, recurrences are not allowed which reduces the 256 possibilities to 4! combinations. From these 24, not all are different from each other because of symmetries in the indices of some second order relations. For (2.86) and eq. (2.97) this means that only $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$ choices remain. They all yield non-trivial results. In eq. (2.88) and in eq. (2.91)[15] 12 cases remain, because they are only symmetric in two indices. The remaining relations cannot be analysed here, because there are only three or two indices to be inserted. Those relations with one or more identical indices in the products of structure constants are not accessible to this choice.

We will basically work with (2.86) in this paper since it is already conclusive (as mentioned above in the purely massive case this relation complies with eq. (2.97) so this relation is confirmed from two independent sectors). Let us compile below, what we have found: (2.86), multiplied by $2m_h m_j$, setting $d = X^1$, $a = W^1$, $h = Z$, $j = Y^1$ we get:

$$\begin{aligned} 0 = &\sum_{j=1}^{6} f_{X^1Y^jW^1} f_{Y^1Y^jZ} \\ &+ \frac{m_Y^2 + m_X^2 - m_W^2}{2m_X^2} \sum_{i=1}^{6} f_{X^iY^1W^1} f_{X^1X^iZ} \\ &+ \frac{m_Y^2 + m_W^2 - m_X^2}{2m_W^2} f_{W^1W^2Z} f_{X^1Y^1W^2} \end{aligned} \tag{10.1}$$

where we have cancelled a factor m_Z^2. We go on with (2.86); taking $d = W^1$, $a = Y^1$, $h = X^1$, $j = Z$ yields:

$$\begin{aligned} 0 = &\sum_{i=1}^{6} f_{X^iY^1W^1} f_{X^1X^iZ} \\ &+ \frac{m_Y^2 + m_X^2 - m_W^2}{2m_Y^2} \sum_{j=1}^{6} f_{Y^1Y^jZ} f_{X^1Y^jW^1} \\ &+ \frac{m_W^2 + m_X^2 - m_Y^2}{2m_W^2} f_{X^1Y^1W^2} f_{W^1W^2Z} \end{aligned} \tag{10.2}$$

and with $d = X^1$, $a = Y^1$, $h = Z$, $j = W^1$:

$$\begin{aligned} 0 = &f_{X^1Y^1W^2} f_{W^1W^2Z} \\ &+ \frac{m_X^2 + m_W^2 - m_Y^2}{2m_X^2} \sum_{i=1}^{6} f_{X^iY^1W^1} f_{X^1X^iZ} \\ &+ \frac{m_Y^2 + m_W^2 - m_X^2}{2m_Y^2} \sum_{j=1}^{6} f_{Y^1Y^jZ} f_{X^1Y^jW^1} \end{aligned} \tag{10.3}$$

which is the last equation that involves only the masses m_X, m_Y and m_W. The remaining relations from (2.86) are the following. Setting $d = Z$, $a = W^1$, $h = X^1$,

[15][2] has in eq. (4.4.42) in the last column a misprint: instead of $+2f_{bcj}f_{ahj}$ one should read $-2f_{bcj}f_{ahj}$, therefore this relation is not different from (4.4.39). Another sign should be adjusted in (4.4.51): on the left hand side there is a total sign change.

$j = Y^1$ gives:

$$\begin{aligned} 0 =& f_{W^1W^2Z} f_{X^1Y^1W^2} \\ &+ \frac{2m_X^2 - m_Z^2}{2m_X^2} \sum_{i=1}^{6} f_{X^iY^1W^1} f_{X^1X^iZ} \\ &+ \frac{2m_Y^2 - m_Z^2}{2m_Y^2} \sum_{j=1}^{6} f_{X^1Y^jW^1} f_{Y^1Y^jZ} \end{aligned} \tag{10.4}$$

where a common factor $m_Y^2 + m_X^2 - m_W^2$ has cancelled. (2.86) with $d = Z$, $a = Y^1$, $h = X^1$, $j = W^1$ gives:

$$\begin{aligned} 0 =& -\sum_{j=1}^{6} f_{Y^1Y^jZ} f_{X^1Y^jW^1} \\ &- \frac{2m_X^2 - m_Z^2}{2m_X^2} \sum_{i=1}^{6} f_{X^iY^1W^1} f_{X^1X^iZ} \\ &- \frac{2m_W^2 - m_Z^2}{2m_W^2} f_{X^1Y^1W^2} f_{W^1W^2Z} \end{aligned} \tag{10.5}$$

and (2.86) with $d = Z$, $a = X^1$, $h = W^1$, $j = Y^1$ leads to:

$$\begin{aligned} 0 =& \sum_{i=1}^{6} f_{X^1X^iZ} f_{X^iY^1W^1} \\ &+ \frac{2m_W^2 - m_Z^2}{2m_W^2} f_{X^1Y^1W^2} f_{W^1W^2Z} \\ &+ \frac{2m_Y^2 - m_Z^2}{2m_Y^2} \sum_{j=1}^{6} f_{X^1Y^jW^1} f_{Y^1Y^jZ}. \end{aligned} \tag{10.6}$$

We see that we have many more relations than structure constants, since the (sums of) structure constants are the same in every relation. Hence it is advisable to eliminate the three (sums of) structure constants

$$\begin{aligned} A &:= \sum_i f_{X^1X^iZ} f_{X^iY^1W^1} \\ B &:= \sum_j f_{Y^1Y^jZ} f_{X^1Y^jW^1} \\ C &:= f_{X^1Y^1W^2} f_{W^1W^2Z} \end{aligned}$$

in the following set of equations

$$\left| \begin{array}{l} 0 = 2Bm_X^2 m_W^2 + A(m_Y^2 + m_X^2 - m_W^2)m_W^2 + C(m_Y^2 + m_W^2 - m_X^2)m_X^2 \\ 0 = 2Am_Y^2 m_W^2 + B(m_Y^2 + m_X^2 - m_W^2)m_W^2 C(m_W^2 + m_X^2 - m_Y^2)m_Y^2 \\ 0 = 2Cm_X^2 m_Y^2 + A(m_X^2 + m_W^2 - m_Y^2)m_Y^2 + B(m_Y^2 + m_W^2 - m_X^2)m_X^2 \\ 0 = 2Cm_Y^2 m_X^2 + A(2m_X^2 - m_Z^2)m_Y^2 + B(2m_Y^2 - m_Z^2)m_X^2 \\ 0 = 2Bm_X^2 m_W^2 + A(2m_X^2 - m_Z^2)m_W^2 + C(2m_W^2 - m_Z^2)m_X^2 \\ 0 = 2Am_Y^2 m_W^2 + C(2m_W^2 - m_Z^2)m_Y^2 + B(2m_Y^2 - m_Z^2)m_W^2 \end{array} \right| . \tag{10.7}$$

All equations are linear in A, B and C. Therefore, an elimination with a computer program like Mathematica poses no problem. Elimination is, of course, only possible if A, B and C are different from zero. We will assume this at first. If C was zero we would be left with a trivial theory, where none of the new gauge bosons would couple to the W^1 (or by interchange W^2). We will later discuss what happens when one or both of the sums A or B are zero. First we do an elimination of A and B using the first three equations of (10.7). From this elimination the restriction

$$\begin{aligned} &Cm_X^2 m_Y^2 (m_W^6 - m_W^4 m_X^2 - m_W^2 m_X^4 + m_X^6 - m_W^4 m_Y^2 + 2m_W^2 m_X^2 m_Y^2 \\ &- m_X^4 m_Y^2 - m_W^2 m_Y^4 - m_X^2 m_Y^4 + m_Y^6) = 0 \end{aligned} \tag{10.8}$$

results with the following solutions:

$$\begin{aligned} C &= 0 \\ m_Y^2 &= 0 \\ m_X^2 &= 0 \\ m_W^2 &= m_X^2 + m_Y^2 \\ m_W^2 &= m_X^2 - m_Y^2 \\ m_W^2 &= -m_X^2 + m_Y^2. \end{aligned} \tag{10.9}$$

Now taking the fourth and the fifth equation in (10.7) and eliminating A we get the following simple restriction

$$Cm_Y^2 - Bm_W^2 = 0 \tag{10.10}$$

or if $m_W^2 \neq 0$

$$B = C\frac{m_Y^2}{m_W^2}. \tag{10.11}$$

From the last two equations in (10.7) we have the additional restriction

$$Cm_Y^2 m_Z^2(-2m_W^2 + m_Z^2) = Bm_W^2(2(m_X^2 + m_Y^2) - m_Z^2)m_Z^2. \tag{10.12}$$

Elimination of B in (10.11) and (10.12) yields

$$Cm_X^2 m_Y^2 (m_W^2 + m_X^2 + m_Y^2 - m_Z^2) m_Z^2 = 0 \tag{10.13}$$

which would mean

$$m_W^2 = m_Z^2 - m_X^2 - m_Y^2; \tag{10.14}$$

but this is in contradiction to eqs . (10.9). We thus see already from this sector alone that with both A and B non-zero, the Georgi-Glashow model with these specific mass degeneracies and charges for the super-heavy bosons is not possible.

What happens if we set $A = 0$? This has no physical implications because it only means that the *sum* vanishes not the individual products of structure constants. We take the fourth and the fifth equation of (10.7) as before and eliminate B getting

$$Cm_X^2(2(m_W^2 + m_Y^2) - m_Z^2)m_Z^2 = 0 \tag{10.15}$$

with the only non-trivial solution

$$m_W^2 = \frac{m_Z^2}{2} - m_Y^2. \tag{10.16}$$

Taking the last two equations in (10.7) for elimination of B we get

$$Cm_X^2 m_Z^2(-2m_W^2 + m_Z^2) = 0 \tag{10.17}$$

with the only non-trivial solution

$$m_W^2 = \frac{m_Z^2}{2} \tag{10.18}$$

and comparing with (10.16) this would lead to $m_Y^2 = 0$ which is not true.

Now we assume $B = 0$ and take the fourth and the fifth equation of (10.7) as before and eliminate A getting

$$Cm_X^2 m_Y^2 m_Z^2 = 0 \tag{10.19}$$

and this leads directly to $C = 0$ which would only be true if none of the new bosons coupled to the W^2. Finally if $A = 0$ *and* $B = 0$ we are again forced to set $C = 0$.

We can also derive no-go results from other sectors separately and in combination, taking equations from different sectors. This means we have *physically* ruled out the Georgi-Glashow model. *Mathematically* there was not yet a contradiction. We are just forced to set some of the variables to zero which is physically not tenable. Trivial solutions with some of the masses or sums of structure constants set to zero always survive.

Remark: Note that (10.7) does also not allow for the Georgi-Glashow $\mathfrak{su}(5)$ model, where $m_X^2 = m_Y^2$, which would mean that only one Higgs multiplet was present. This is often used as a first approximation.

10.1. **Other mass degeneracies, no-go Results.** We consider in the following the $\mathfrak{su}(5)$ model with less mass degeneracy for the super-heavy gauge bosons X and Y. The charges for X and Y remain unchanged. This too leads to no-go results. First let us assume that per two X particles the mass is equal, i.e. $m_{X^1} = m_{X^2} \neq m_{X^3} = m_{X^4} \neq m_{X^5} = m_{X^6}$ and similarly for the Y bosons.

We have thus the following field content:

(1) 8 massless gluons
(2) 1 massless photon
(3) 2 massive charged W bosons with mass m_W
(4) 1 massive uncharged Z^0 with mass m_Z
(5) 6 massive charged X bosons with masses $m_i,\ i = 1, 3, 5$
(6) 6 massive charged Y bosons with masses $m_j,\ j = 1, 3, 5$

where we write m_1 for m_{X^1} and m_{X^2} etc. when there is no confusion, i.e. when not both the X and Y particles are appearing together. It is now to be investigated, whether such a theory is conformable with second order gauge invariance.

Let us first go to (2.89), setting $a = X^1$, $b = X^2$ and $c = X^3$ and with the masses $m_{X^1} = m_{X^2} \neq m_{X^3}$. For the subscript X^i we will write just the index i for short, leading to

$$0 = -m_3^2 \sum_{d>r} f_{12d} f_{23d} + \sum_{j=1}^{r} \frac{1}{4}\big[3m_j^2 - 3m_1^2 - m_3^2\big] f_{12j} f_{23j}. \tag{10.20}$$

Note that the first sum over the massless indices is zero due to eq. (2.90), because its left hand side is not zero in our case

$$(m_1^2 - m_3^2) \sum_{d>r} (f_{23d})^2 = 0. \tag{10.21}$$

Hence f_{23d} must vanish. As seen in section 9.1 the only possibility for a massive boson to couple to two X bosons, say X^1 and X^2, is the Z^0 boson. Therefore, (10.20) reduces to

$$0 = \frac{1}{4}[3m_Z^2 - 3m_1^2 - m_3^2]f_{12Z}f_{23Z} \tag{10.22}$$

and in the case that both f_{12Z} and f_{23Z} are not zero we could deduce the relation

$$3m_1^2 + m_3^2 = 3m_Z^2. \tag{10.23}$$

Such a relation is not tenable for a theory with super-heavy masses m_1 and m_3. We are therefore required to set one or both structure constants to zero. We first discuss the case when $f_{12Z} \neq 0$ then $f_{23Z} \neq 0$ and in the end $f_{12Z} = 0$ *and* $f_{23Z} = 0$.

Case $f_{12Z} \neq 0$. In this case $f_{23Z} = 0$ ($c = 3$) and by setting the index $c = 4$ we have the same relation (10.20), as $m_3 = m_4$ leading to $f_{24Z} = 0$. In the same line of reasoning we have finally for $c = 5, 6$

$$3m_1^2 + m_5^2 = 3m_Z^2 \tag{10.24}$$

leading to $f_{25Z} = 0$ and $f_{26Z} = 0$. Note that $c = b$ is not allowed in this relation and that is why we did not set $c = 2$ in this case.

Now we interchange the index a and b in eq. (2.89) giving

$$0 = \frac{1}{4}[3m_Z^2 - 3m_3^2 - m_1^1]f_{21Z}f_{13Z} \tag{10.25}$$

with f_{21Z} still different from zero due to antisymmetry. We deduce in the same manner: $f_{1cZ} = 0$ with varying c as above. In the following table we summarise what we have achieved so far; the second column refers to a and b interchanged in (2.89):

$$\begin{array}{c|c|c} f_{12Z} \neq 0 & f_{23Z} = 0 & f_{13Z} = 0 \\ & f_{24Z} = 0 & f_{14Z} = 0 \\ & f_{25Z} = 0 & f_{15Z} = 0 \\ & f_{26Z} = 0 & f_{16Z} = 0 \end{array}$$

The next structure constant with one index equal to 3 that could be different from zero is f_{34Z} and assuming it to be different from zero[16] we have

$$\begin{array}{c|c|c} f_{34Z} \neq 0 & f_{45Z} = 0 & f_{35Z} = 0 \\ & f_{46Z} = 0 & f_{36Z} = 0 \\ & f_{41Z} = 0 & f_{31Z} = 0 \\ & f_{42Z} = 0 & f_{32Z} = 0 \end{array}$$

and assuming $f_{56Z} \neq 0$ we have

$$\begin{array}{c|c|c} f_{56Z} \neq 0 & f_{61Z} = 0 & f_{51Z} = 0 \\ & f_{62Z} = 0 & f_{52Z} = 0 \\ & f_{63Z} = 0 & f_{53Z} = 0 \\ & f_{64Z} = 0 & f_{54Z} = 0 \end{array}$$

[16] The arguments remain also true if we make an other choice, e.g. if $f_{35Z} \neq 0$ only f_{46Z} can be different from zero, all others have to vanish.

where the last column corresponds again to a and b interchanged in eq. (2.89). As a result we have now three structure constants that couple the Z^0 particle to two massive X's. We have to see whether this goes along with our other second order relations. Setting $a = 1$, $b = 3$, $d = 2$ and $j = 4$ in eq. (2.97) we have specified this relation in the same way as eq. (2.89) above with $m_1 = m_2 \neq m_3 = m_4$ but now with only three possible couplings for Z^0 to the X bosons

$$\begin{aligned} 0 =& \sum_{c>r} f_{13c} f_{2c4} \frac{m_3^2 + m_1^2}{2m_3} + f_{13Z} f_{2Z4} \frac{m_3^2 - m_Z^2 + m_1^2}{2m_3} \\ &+ f_{14Z} f_{23Z} \frac{m_Z^2 + m_3^2 - m_1^2}{4m_3 m_Z^2} (m_Z^2 - m_3^2 + m_1^2) \\ &- f_{34Z} f_{21Z} \frac{m_Z^2 + m_3^2 - m_3^2}{4m_3 m_Z^2} (m_Z^2 - m_1^2 + m_1^2) \\ =& - f_{34Z} f_{21Z} \frac{m_Z^2}{4m_3}. \end{aligned} \tag{10.26}$$

We first note that $f_{13c} = 0$ for the massless indices c according to (2.90). Then due to our reasoning above also $f_{13c} = 0$ and $f_{14c} = 0$ holds for massive indices i.e. the coupling to Z^0. The last equality produces again a result that is not tenable for a reasonable gauge theory. One or both of the structure constants must be zero then. If f_{12Z} is not zero we have $f_{34Z} = 0$ and with the same reasoning ($b = 5, j = 6$) $f_{56Z} = 0$. If both f_{12Z} and f_{34Z} vanish, we cannot constrain f_{56Z}: In any case we thus have only one coupling that survives.

Let us try this last possibility with only one coupling from Z^0 to two X's assuming f_{12Z} not to be zero. In (2.86) we find, setting $a = W^1$, $d = W^2$, $h = X^1$ and $j = X^2$, a relation that is not contradictory (including couplings to the Y bosons) but the special case (2.87) gives a further strong constraint demanding a symmetry in the coupling of the different X's: Setting $j = X^1$ (index 1), $h = X^2$ we find with $m_1 = m_2$

$$\begin{aligned} \sum_{p=1}^{t} f_{11p}^5 f_{11p}^5 &= \frac{1}{2m_1^2} \left[\sum_{c=1}^{r+s} (2m_1^2 - m_c^2) f_{12c} f_{12c} - \frac{m_Z^2}{2} f_{12Z} f_{12Z} \right] \\ &= 1 \sum_{c>r} f_{12c} f_{12c} + \left(1 - \frac{m_Z^2}{2m_1^2} - \frac{m_Z^2}{4m_1^2}\right) f_{12Z} f_{12Z} \\ &= \sum_{c=1}^{r+s} f_{12c} f_{12c} - \frac{3}{4} \frac{m_Z^2}{m_1^2} f_{12Z} f_{12Z} \\ &\neq 0 \end{aligned} \tag{10.27}$$

As in the derivation of the electroweak theory we use here again the fact that the left hand side of (10.27) remains the same while changing the index h. Let us set $h = X^3$ with now $m_1 \neq m_3$

$$\begin{aligned} \sum_{p=1}^{t} f_{11p}^5 f_{11p}^5 &= \frac{1}{2m_1^2} \left[\sum_{c=1}^{r+s} (m_1^2 + m_3^2 - m_c^2) f_{13c} f_{13c} - \frac{m_Z^4 - (m_1^2 - m_3^2)^2}{2m_Z^2} f_{13Z} f_{13Z} \right] \\ &= 0 \end{aligned} \tag{10.28}$$

because f_{13c} is zero with c standing for both massive and massless indices. We therefore see that one coupling alone leads to a contradiction.

Case $f_{23Z} \neq 0$. If $f_{23Z} \neq 0$ we see this time varying the index a in eq. (2.89) that all structure constants of the form f_{a2Z} and, by interchanging b and c which is possible because the mass factor does not change, also those of the form f_{a3Z} must vanish: and assuming $f_{14Z} \neq 0$ we have

$$f_{23Z} \neq 0 \left| \begin{array}{l} f_{12Z} = 0 \\ f_{42Z} = 0 \\ f_{52Z} = 0 \\ f_{62Z} = 0 \end{array} \right| \begin{array}{l} f_{13Z} = 0 \\ f_{43Z} = 0 \\ f_{53Z} = 0 \\ f_{63Z} = 0, \end{array}$$

implying that all structure constants of the form and f_{4cZ} with and f_{1cZ} with $c = 2, 3, 5, 6$ must vanish. So we have only f_{56Z} as non-vanishing structure constant left. With these three remaining structure constants we go through the same line of reasoning as in the case $f_{12Z} \neq 0$ above and get a contradiction.

Case $f_{12Z} = 0$ and $f_{23Z} = 0$. If both structure constants vanish, we could start again with say f_{13Z} and f_{3cZ} in the same way as in the other cases. If then one structure constant is chosen to be different from zero only three non-vanishing structure constants survive in the same way as in the other cases or we would have the trivial solution with all structure constants zero.

10.2. **A last-ditch attempt.** Finally we discuss other possible mass degeneracies for the heavy bosons X and Y. First let us assume that per three X particles the mass is equal, i.e. $m_{X^1} = m_{X^2} = m_{X^3} \neq m_{X^4} = m_{X^5} = m_{X^6}$ and similarly for the Y bosons. That means this time the field content is:

(1) 8 massless gluons
(2) 1 massless photon
(3) 2 massive charged W bosons with mass m_W
(4) 1 massive uncharged Z^0 with mass m_Z
(5) 6 massive charged X bosons with masses $m_i,\ i = 1, 2$
(6) 6 massive charged Y bosons with masses $m_j,\ j = 1, 2$.

We have now two possibilities in eq. (2.89), one with only one mass m_X in the mass factor, setting $a = 1,\ b = 2,\ c = 3$ or $a = 4,\ b = 5,\ c = 6$ or one with two different masses in the mass factor m_{X^1} and m_{X^2}, setting e.g. $a = 1,\ b = 2$ but $c = 4, 5$ or 6. The case of one common mass we have treated above. We consider therefore only the second alternative with two masses, but this turns out to lead to the same relations as discussed in section 10.1 with three masses for the six X or Y bosons. Also here one can reduce the number of non-vanishing structure constants to three. Finally the last step as in (10.27) is as well recovered by setting in eq. (2.87) $j = 1$ and $h = 2$ and $h = 4$ respectively.

11. General discussion for $\mathfrak{A}(4)$ and beyond

We have seen that one specific model for grand unification is ruled out by gauge invariance. This could have two reasons

– The arguments where based on assumptions on the charges of the new bosons and the coupling structure was thus selective.

- According to the Higgs potential we assumed four distinct masses for the model; a change could lead to an other mass spectrum, than the assumed so far.

In this section we will work in a framework compatible to $\mathfrak{A}(4)$ as the Lie algebra but we will no more make assumptions on the charges and the mass spectrum of the new heavy particles as in $\mathfrak{su}(5)$. As a consequence our coupling structure is more general and we will deal with more couplings that are in principle different from zero.

We will see that gauge invariance is then not strong enough to entirely determine the model (whether $\mathfrak{su}(5)$ or beyond) but we find restrictions on the masses and on the structure constants which are in principle applicable to the algebra $\mathfrak{A}(4)$ as one considers 24 gauge bosons again or beyond by letting the sums run over even more generators. Yet it is not clear at all whether the algebra must be simple, it might be, that the solutions allow only for a reductive algebra. We will therefore investigate the physical order of magnitude of the products of coupling constants. Such consideration though are based upon specific gauge hierarchies and can thus not be conclusive in the general case. In the special case of the gauge hierarchy in Georgi-Glashow $\mathfrak{su}(5)$, we do not find structure constants of the same order which is again a hint that the model is too simple.

In the following we give all relevant gauge restrictions from the different sectors and stick to the following manner-of-speaking:

- *Off-diagonal* restrictions are those where one inserts four different indices in the product of structure constants i.e. in $f_{12c}f_{34c}$ all indices are different. The masses can be degenerate of course.
- *Semi-diagonal* we will call those restrictions where only three indices are different (one common index).
- *Diagonal* are those with only two different indices 1 and 2, as in f_{12c}^2. In this case the Higgs-couplings are different from zero.

The sum over c will be split later into summands within the same multiplet i.e. with mass degeneracy. A mass degeneracy always means a factorisation and we are left with one parameter only. The question is, how many such summands we can determine with our gauge restrictions. We have seen this already in the case of the Georgi-Glashow model. For $\mathfrak{su}(5)$ we would have to split c into sums over five distinct masses $m_\gamma = m_\lambda = 0$, m_W, m_Z, m_X and m_Y in each sum:

$$\begin{aligned}\sum_c f_{12c}f_{34c} &= \sum_d f_{12d}f_{34d} + \sum_j f_{12j}f_{34j} \\ &= \sum_d f_{12d}f_{34d} + f_{12Z}f_{34Z} \sum_{i=1,2} f_{12W^i}f_{34W^i} \\ &\quad + \sum_i f_{12X^i}f_{34X^i} + \sum_i f_{12Y^i}f_{34Y^i} + \dots \\ &=: f[d] + f[Z] + f[W] + f[X] + f[Y] + \dots\end{aligned} \tag{11.1}$$

where d runs over massless indices only. Apart from two new masses m_X and m_Y, we can add other sums of couplings with different masses.

11.1. **Off-diagonal gauge restrictions.** Dependent on the symmetry of the relation with regard to its indices we have a different number of equations. For (2.86) this means 6, for (2.88) and (2.91) there are 12. Since in the special cases it is not possible to choose 4 indices, we have a total of 30 equations.[17] We write them here in a form that is easily adapted to a Mathematica or Maple input: The squared masses we put as

$$m[X] := m_X^2, \quad m[W] := m_W^2$$

and so on. The product of structure constants we denote as:

$$\begin{aligned} f[d] &:= \sum_d f_{12d} f_{34d} \\ g[d] &:= \sum_d f_{13d} f_{24d} \\ h[d] &:= f_{14d} f_{23d}. \end{aligned}$$

For the couplings to the massless bosons d and for the massive j we write:

$$\begin{aligned} f[j] &:= \sum_j f_{12j} f_{34j} \\ g[j] &:= \sum_j f_{13j} f_{24j} \\ h[j] &:= \sum_j f_{14j} f_{23j}. \end{aligned}$$

The relations from (2.86) are:

$$\begin{aligned} &\sum_d f[d](m[4] + m[3]) + \sum_j f[j](m[4] + m[3] - m[j]) \\ &- \sum_j g[j]((m[j] + m[4] - m[2])(m[j] + m[3] - m[1])/(2m[j])) \\ &+ \sum_j h[j]((m[j] + m[3] - m[2])(m[j] + m[4] - m[1])/(2m[j])) = 0, \end{aligned} \tag{11.2}$$

$$\begin{aligned} &\sum_d g[d](m[4] + m[2]) + \sum_j g[j](m[4] + m[2] - m[j]) \\ &- \sum_j f[j]((m[j] + m[4] - m[3])(m[j] + m[2] - m[1])/(2m[j])) \\ &- \sum_j h[j]((m[j] + m[2] - m[3])(m[j] + m[4] - m[1])/(2m[j])) = 0, \end{aligned} \tag{11.3}$$

$$\begin{aligned} &\sum_d h[d](m[3] + m[2]) + \sum_j h[j](m[3] + m[2] - m[j]) \\ &+ \sum_j f[j]((m[j] + m[3] - m[4])(m[j] + m[2] - m[1])/(2m[j])) \\ &- \sum_j g[j]((m[j] + m[2] - m[4])(m[j] + m[3] - m[1])/(2m[j])) = 0, \end{aligned} \tag{11.4}$$

[17] The 12 equations from (2.91) are a consistency check here, we do not list them below.

$$\begin{aligned}
&\sum_d h[d](m[1]+m[4]) + \sum_j h[j](m[1]+m[4]-m[j]) \\
&+ \sum_j f[j]((m[j]+m[4]-m[3])(m[j]+m[1]-m[2])/(2m[j])) \\
&- \sum_j g[j]((m[j]+m[1]-m[3])(m[j]+m[4]-m[2])/(2m[j])) = 0,
\end{aligned} \tag{11.5}$$

$$\begin{aligned}
&\sum_d g[d](m[1]+m[3]) + \sum_j g[j](m[1]+m[3]-m[j]) \\
&- \sum_j f[j]((m[j]+m[3]-m[4])(m[j]+m[1]-m[2])/(2m[j])) \\
&- \sum_j h[j]((m[j]+m[1]-m[4])(m[j]+m[3]-m[2])/(2m[j])) = 0,
\end{aligned} \tag{11.6}$$

$$\begin{aligned}
&\sum_d f[d](m[1]+m[2]) + \sum_j f[j](m[1]+m[2]-m[j]) \\
&- \sum_j g[j]((m[j]+m[2]-m[4])(m[j]+m[1]-m[3])/(2m[j])) \\
&+ \sum_j h[j]((m[j]+m[1]-m[4])(m[j]+m[2]-m[3])/(2m[j])) = 0,
\end{aligned} \tag{11.7}$$

The relations from (2.88) read as follows:

$$\begin{aligned}
&- \sum_d g[d]m[4] - \sum_d h[d]m[3] + \sum_j f[j]((m[j]+m[2]-m[1])(m[4]-m[3])/(2m[j])) \\
&+ \sum_j h[j]((m[j]-m[3]) - (m[j]+m[2]-m[3])(m[j]+m[1]-m[4])/(4m[j])) \\
&+ \sum_j g[j]((m[j]-m[4]) - (m[j]+m[2]-m[4])(m[j]+m[1]-m[3])/(4m[j])) = 0,
\end{aligned} \tag{11.8}$$

$$\begin{aligned}
&- \sum_d f[d]m[4] + \sum_d h[d]m[2] + \sum_j g[j]((m[j]+m[3]-m[1])(m[4]-m[2])/(2m[j])) \\
&- \sum_j h[j]((m[j]-m[2]) - (m[j]+m[3]-m[2])(m[j]+m[1]-m[4])/(4m[j])) \\
&+ \sum_j f[j]((m[j]-m[4]) - (m[j]+m[3]-m[4])(m[j]+m[1]-m[2])/(4m[j])) = 0,
\end{aligned} \tag{11.9}$$

$$\begin{aligned}
&\sum_d f[d]m[3] + \sum_d g[d]m[2] + \sum_j h[j]((m[j]+m[4]-m[1])(m[3]-m[2])/(2m[j])) \\
&- \sum_j g[j]((m[j]-m[2]) - (m[j]+m[4]-m[2])(m[j]+m[1]-m[3])/(4m[j])) \\
&- \sum_j f[j]((m[j]-m[3]) - (m[j]+m[4]-m[3])(m[j]+m[1]-m[2])/(4m[j])) = 0,
\end{aligned} \tag{11.10}$$

$$-\sum_d h[d]m[4]-\sum_d g[d]m[3]-\sum_j f[j]((m[j]+m[1]-m[2])(m[4]-m[3])/(2m[j]))$$
$$+\sum_j g[j]((m[j]-m[3])-(m[j]+m[1]-m[3])(m[j]+m[2]-m[4])/(4m[j]))$$
$$+\sum_j h[j]((m[j]-m[4])-(m[j]+m[1]-m[4])(m[j]+m[2]-m[3])/(4m[j]))=0, \tag{11.11}$$

$$\sum_d f[d]m[4]+\sum_d g[d]m[1]+\sum_j h[j]((m[j]+m[3]-m[2])(m[4]-m[1])/(2m[j]))$$
$$-\sum_j g[j]((m[j]-m[1])-(m[j]+m[3]-m[1])(m[j]+m[2]-m[4])/(4m[j]))$$
$$-\sum_j f[j]((m[j]-m[4])-(m[j]+m[3]-m[4])(m[j]+m[2]-m[1])/(4m[j]))=0, \tag{11.12}$$

$$\sum_d h[d]m[1]-\sum_d f[d]m[3]-\sum_j g[j]((m[j]+m[4]-m[2])(m[1]-m[3])/(2m[j]))$$
$$+\sum_j f[j]((m[j]-m[3])-(m[j]+m[4]-m[3])(m[j]+m[2]-m[1])/(4m[j]))$$
$$-\sum_j h[j]((m[j]-m[1])-(m[j]+m[4]-m[1])(m[j]+m[2]-m[3])/(4m[j]))=0, \tag{11.13}$$

$$\sum_d h[d]m[4]-\sum_d f[d]m[2]-\sum_j g[j]((m[j]+m[1]-m[3])(m[4]-m[2])/(2m[j]))$$
$$+\sum_j f[j]((m[j]-m[2])-(m[j]+m[1]-m[2])(m[j]+m[3]-m[4])/(4m[j]))$$
$$-\sum_j h[j]((m[j]-m[4])-(m[j]+m[1]-m[4])(m[j]+m[3]-m[2])/(4m[j]))=0, \tag{11.14}$$

$$\sum_d g[d]m[4]+\sum_d f[d]m[1]-\sum_j h[j]((m[j]+m[2]-m[3])(m[4]-m[1])/(2m[j]))$$
$$-\sum_j f[j]((m[j]-m[1])-(m[j]+m[2]-m[1])(m[j]+m[3]-m[4])/(4m[j]))$$
$$-\sum_j g[j]((m[j]-m[4])-(m[j]+m[2]-m[4])(m[j]+m[3]-m[1])/(4m[j]))=0, \tag{11.15}$$

$$-\sum_d g[d]m[2]-\sum_d h[d]m[1]+\sum_j f[j]((m[j]+m[4]-m[3])(m[2]-m[1])/(2m[j]))$$
$$+\sum_j h[j]((m[j]-m[1])-(m[j]+m[4]-m[1])(m[j]+m[3]-m[2])/(4m[j]))$$
$$+\sum_j g[j]((m[j]-m[2])-(m[j]+m[4]-m[2])(m[j]+m[3]-m[1])/(4m[j]))=0, \tag{11.16}$$

$$\sum_d g[d]m[3]+\sum_d f[d]m[2]-\sum_j h[j]((m[j]+m[1]-m[4])(m[3]-m[2])/(2m[j]))$$
$$-\sum_j f[j]((m[j]-m[2])-(m[j]+m[1]-m[2])(m[j]+m[4]-m[3])/(4m[j]))$$
$$-\sum_j g[j]((m[j]-m[3])-(m[j]+m[1]-m[3])(m[j]+m[4]-m[2])/(4m[j]))=0, \tag{11.17}$$

$$\sum_d h[d]m[3]-\sum_d f[d]m[1]-\sum_j g[j]((m[j]+m[2]-m[4])(m[3]-m[1])/(2m[j]))$$
$$+\sum_j f[j]((m[j]-m[1])-(m[j]+m[2]-m[1])(m[j]+m[4]-m[3])/(4m[j]))$$
$$-\sum_j h[j]((m[j]-m[3])-(m[j]+m[2]-m[3])(m[j]+m[4]-m[1])/(4m[j]))=0, \tag{11.18}$$

$$-\sum_d h[d]m[2]-\sum_d g[d]m[1]-\sum_j f[j]((m[j]+m[3]-m[4])(m[2]-m[1])/(2m[j]))$$
$$+\sum_j g[j]((m[j]-m[1])-(m[j]+m[3]-m[1])(m[j]+m[4]-m[2])/(4m[j]))$$
$$+\sum_j h[j]((m[j]-m[2])-(m[j]+m[3]-m[2])(m[j]+m[4]-m[1])/(4m[j]))=0. \tag{11.19}$$

The relations from the last sector (2.97) give in the purely massive case the same restrictions as (2.86). Note that in special cases these relations are not all independent.

11.2. **Semi-diagonal gauge restrictions.** Now we come to the case with one common index. The notation here means

$$f_{\mathrm{sd}}[d] := \sum_d f_{12d}f_{24d}$$
$$f_{\mathrm{sd}}[j] := \sum_j f_{12f}f_{24j}.$$

One has to be careful here since in the relations not any two indices can be equal. In (2.86) one cannot choose $j = h$[18] and we find only two relations that can be different from each other:

[18]With the input $j = h$ the relations are in this case trivially fulfilled, but in some cases one gets wrong results. That is why we have to treat semi-diagonal and diagonal cases separately.

$$\sum_d f_{\mathrm{sd}}[d](m[2]+m[4]) + \sum_j f_{\mathrm{sd}}[j]((m[4]+m[2]-m[j])-$$
$$(m[j]+m[4]-m[2])(m[j]+m[2]-m[1])/(2m[j])) = 0, \quad (11.20)$$

$$\sum_d f_{\mathrm{sd}}[d](m[1]+m[2]) + \sum_j f_{\mathrm{sd}}[j]((m[1]+m[2]-m[j])-$$
$$(m[j]+m[2]-m[4])(m[j]+m[1]-m[2])/(2m[j])) = 0, \quad (11.21)$$

In (2.88) we have $b \neq c$ and get

$$-\sum f_{\mathrm{sd}}[d]m[4] + \sum_j f_{\mathrm{sd}}[j]((m[j]-m[4]) + (m[j]+m[2]-m[1])(m[4]-m[2])/(2m[j])$$
$$-(m[j]+m[2]-m[4])(m[j]+m[1]-m[2])/(4m[j])) = 0, \quad (11.22)$$

$$-\sum_d f_{\mathrm{sd}}[d]m[2] + \sum_j f_{\mathrm{sd}}[j]((m[j]-m[2]) - (m[j]+m[1]-m[2])(m[4]-m[2])/(2m[j])$$
$$-(m[j]+m[2]-m[4])(m[j]+m[1]-m[2])/(4m[j])) = 0, \quad (11.23)$$

$$\sum_d f_{\mathrm{sd}}[d](m[4]+m[1]) + \sum_j f_{\mathrm{sd}}[j](-(m[j]-m[4]) - (m[j]-m[1])$$
$$+(m[j]+m[2]-m[1])(m[j]+m[2]-m[4])/(2m[j])) = 0, \quad (11.24)$$

$$-\sum_d f_{\mathrm{sd}}[d]m[1] + \sum_j f_{\mathrm{sd}}[j]((m[j]-m[1]) - (m[j]+m[2]-m[4])(m[2]-m[1])/(2m[j])$$
$$-(m[j]+m[2]-m[1])(m[j]+m[4]-m[2])/(4m[j])) = 0, \quad (11.25)$$

$$-\sum_d f_{\mathrm{sd}}[d]m[2] + \sum_j f_{\mathrm{sd}}[j]((m[j]-m[2]) - (m[j]+m[4]-m[2])(m[1]-m[2])/(2m[j])$$
$$-(m[j]+m[4]-m[2])(m[j]+m[2]-m[1])/(4m[j])) = 0. \quad (11.26)$$

One can, as a check, see whether the other special relations (2.89), (2.91) add new relations. We will though not list them here. Only the case $b = c$ could yield additional restrictions. From (2.94) we have

$$-\big(\sum_d f_{\mathrm{sd}}[d]m[2] + \sum_j f_{\mathrm{sd}}[j]((m[j]-m[2])(-3m[j]+m[1]-m[2]+m[4])$$
$$+m[1]m[4])/(4m[j])\big) = 0. \quad (11.27)$$

Here we will make later the same split into summands as indicated in (11.1).

11.3. **Diagonal: Restrictions with Higgs-couplings.** Here we obtain relations between the Higgs-couplings and the Yang-Mills structure constants. One can eliminate the Higgs couplings from the equations, gaining new restrictions for the f_{abc}'s alone. With

$$f_d[d] := \sum_d f_{12d} f_{12d}$$
$$f_d[j] := f_{12j} f_{12j}$$

and H denoting the sum over the Higgs-couplings f^5, we find from (2.87)

$$H = \sum_d f_d[d]^2(m[1]+m[2])/(2m[2]) + \sum_j f_d[j]^2((m[1]+m[2]-m[j])/(2m[2])-$$
$$(m[j]^2-(m[1]-m[2])^2)/(4m[j]m[2])), \tag{11.28}$$

from (2.93)

$$\begin{aligned} H = \sum_d f_d[d]^2(m[1]/m[2]) + \sum_j f_d[j]^2((m[j]-m[1])(-3m[j]-m[1] \\ + 2m[2]) + m[2]^2)/(4m[j]m[2]), \end{aligned} \tag{11.29}$$

from (2.89)

$$\begin{aligned} H = \sum_d f_d[d]^2 m[1]/m[2] - \sum_j f_d[j]^2((m[j]-m[2])(3m[j]-m[1]+m[2]) \\ - m[1](m[j]+m[1]-m[2]))/(4m[j]m[2]) \end{aligned} \tag{11.30}$$

and from (2.92)

$$\begin{aligned} H = \sum_d f_d[d]^2 m[1]/(m[2]) + \sum_j f_d[j]^2((m[1]-m[j]) \\ + ((m[j]+m[2]-m[1])^2)/(4m[j]))/(m[2]). \end{aligned} \tag{11.31}$$

If one eliminates the sum over the Higgs couplings one obtains as restriction exactly (2.90). From the diagonal couplings no more information on the f_{abc} can be gained.

12. Evaluation of the systems of equations

The systematics of the treatment of the general systems of equations is the following: As in the Georgi-Glashow model we will treat the products of structure constants as linear parameters in our second order relations. In this case a Gauss elimination can be done without ambiguities. In order to demonstrate our proceeding we show it first for a smaller theory that has already be treated in [26]. Then we look at the Georgi-Glashow model again. Finally we add more couplings to this model and look for all possible restrictions.

Let us make here a remark on two extreme cases: Due to (2.90) massive gauge bosons can only couple to massless ones when their masses are equal. Some mass degeneracy among the new particles is therefore required, a theory without mass degeneracy is trivial. The other extreme is when all new gauge bosons have the same mass. In this case couplings from massless to massive indices are free. Only couplings among the W's and the gluons are forbidden. We will in the following have a special look on this degenerate case.

12.1. **Electroweak theory with a massive photon.** We want briefly repeat in our notation the study in [26] where the electroweak theory with a massive photon has already been considered. In order to show that there is no such model, [26] uses one of the semi-diagonal restrictions and looks for all combinations of structure constants in the model. This is possible because in that model there are only four possibly non-vanishing structure constants for the four gauge bosons $1, 2, 3$ and 4

$$f_{123}, f_{124}, f_{134} \text{ and } f_{234}. \tag{12.1}$$

In case only one of those four structure constants is different from zero, the theory is trivial. In case of two non-vanishing structure constants, one derives relations for the masses of the gauge bosons, that lead again to a trivial theory. The last case, when all four structure constants are non-zero, can be seen (in our notation) as follows. We can e.g. look at the product

$$f_{\mathrm{sd}}[3] = f_{123}f_{243} = -f_{123}f_{234}$$

with the common index 2 which corresponds to $f_{\mathrm{sd}}[d]$ when index 3 is massless or else to $f_{\mathrm{sd}}[[j]$, as it must be here. That is, we have only *one* product in our relations. Since our linear system has in this case rank 2, we are able to determine two products in terms of the masses of the gauge bosons, if a non-trivial solution exists, as can be seen in table 2. Both $f[d]$ and $f[j]$ cannot be different from zero here because

Product	Rank	Solutions
$f_{\mathrm{sd}}[d] = 0,\ \sum_j f_{\mathrm{sd}}[j] = f_{\mathrm{sd}}[3] \neq 0$	2	no solution
$f_{\mathrm{sd}}[d] \neq 0,\ f_{\mathrm{sd}}[3] = 0$	1	cannot occur
$f_{\mathrm{sd}}[d] \neq 0,\ f_{\mathrm{sd}}[3] \neq 0$	3	cannot occur

Table 2. There are no non-trivial solutions for an electroweak theory with a massive photon. Since all indices are massive, we have only one product of structure constants. The off-diagonal restrictions are not applicable here.

there is only one index left; case 2 and 3 in [26] can be treated similarly. The result is of no surprise, but helps to understand the procedure with the (semi-diagonal) restrictions in the following. Next we look again at the Georgi-Glashow model.

12.2. **Retrieving Georgi-Glashow SU(5).** We make here a link from our general system of equations (off-diagonal case) with our system (10.7) above. If we specify the indices to specific massive gauge bosons in the first six equations we arrive again at (10.7), e.g. if $1 = Z$, $2 = W^1$, $3 = X^1$ and $4 = Y^1$, as in (10.4) this means for the structure constants

$$\begin{aligned} f[j] &= f_{12j}f_{34j} = f_{ZW^1W^2}f_{X^1Y^1W^2} = C \\ g[j] &= f_{13j}f_{24j} = \sum_j f_{ZX^1X^j}f_{W^1Y^1X^j} = -A \\ h[j] &= f_{14j}f_{23j} = \sum_j f_{ZY^1Y^j}f_{W^1X^1Y^j} = B \end{aligned} \tag{12.2}$$

and we see that the first equation (11.2) in our general system is equal to the fourth in (10.7). In this way the first six equations (11.2) – (11.7) correspond to (10.7) and we are led to the same conclusions.

What can we say about the semi-diagonal restrictions? Here one cannot choose the indices in a way to obtain in one product the factor $f_{W^1W^2Z}$, because the other factor is then of the form f_{WXZ} which is zero. We look instead whether there can be products of the form $f_{X^1X^2j}f_{X^2X^3j}$ that are not zero. We find a rank 3 linear system with again no non-trivial solutions. The semi-diagonal restrictions can also be used to show that for exotic products like $f_{12j}f_{24j} = f_{ZX^1j}f_{X^1Y^1j}$ there is no solution. This helped us to confine the coupling structure in sect. 9.

Product	Rank	Solutions
$f[d] = g[d] = h[d] = 0$, $\sum_j f_[j], g[j], h[j] \neq 0$	3	no solution
$f_{\rm sd}[d] \neq 0$, $\sum_j f_{\rm sd}[j] \neq 0$	3	"
$f_{\rm sd}[d] = 0$, $\sum_j f_{\rm sd}[j] = f_{\rm sd}[X] \neq 0$	2	"

Table 3. No-go result from off-diagonal and semi-diagonal linear systems for the Georgi-Glashow model.

13. Adding more couplings

We go now a step further and allow for one more coupling in the off-diagonal and in the semi-diagonal equations which means a split in the sums of products f, g or h etc.[19] One would expect the rank of the resulting system to increase but we still find rank 3 for this more complicated coupling scheme! If one includes couplings to massless gauge bosons the system has rank 6 allowing the determination of six products of structure constants. Since we have now 7 unknown parameters we normalise one of them by setting it to one and solve for the others. The mathematical solution for the structure constants can be seen in appendix C. In the off-diagonal case there are qualitatively two types of solutions; dependent on mass degeneracies of the four gauge bosons one chooses for the indices $1, \ldots, 4$, namely

$$\text{Off-diagonal case} \begin{cases} \text{Linear system of rank 6} \\ \left.\begin{array}{ll} m_1 \neq m_2 & m_3 \neq m_4 \\ m_1 \neq m_2 & m_3 = m_4 \\ m_1 = m_2 & m_3 \neq m_4 \end{array}\right\} \text{First solution} \\ \text{Linear system of rank 5} \\ \quad m_1 = m_2 \quad m_3 = m_4 \quad \text{Second solution.} \end{cases} \tag{13.1}$$

In the first solution all couplings from the four massive gauge bosons to massless gauge bosons become zero, as it must be, due to (2.90). For the second solution we find that $f_{12d} f_{34d} \neq 0$ which is a first consistency check. In the following, 13.1 and below, we will look for physical solutions of the general results in appendix C.

What happens to the semi-diagonal relations? Up to now we find that the equations originating from semi-diagonal restrictions where extremely restrictive. As a surprise we found that the rank of the equation system does not increase when adding one additional coupling $f_{\rm sd}[6]$, i.e. adding a gauge boson with a new mass m_6 (or some gauge bosons with a degenerate mass m_6), so that

$$f_{12c} f_{24c} = f_{\rm sd}[d] + f_{\rm sd}[3] + f_{\rm sd}[5] + f_{\rm sd}[6]. \tag{13.2}$$

The indices 3, 5, and 6 stand for particles different from 1,2 and 4 and $m[3] \neq m[5] \neq m[6]$. One can have mass degeneracies among the bosons 1, 2, and 4; the rank remains 3. Only degeneracies between 3, 5, and 6 are not possible. Such a degeneracy would lead to less products because of factorisation and finally allow for trivial solutions only. We look therefore only for solutions with at least four products and find that the semi-diagonal products of structure constants are left

[19] This is a generalisation in the Georgi-Glashow model: it would mean a split e.g. in the sum $h[j] = B = B_1 + B_2$ corresponding to one different mass that couples to the same particles and is possible if one does not restrict the couplings with arguments based on charge conservation.

undetermined, as long as one has at least *three* non-vanishing sums of products of structure constants. With less than three there is no solution. This holds true independent of mass degeneracies — an astonishing result which we summarise in table 4.

Products	Rank	Solutions
$f[d]=g[d]=h[d]=0,\ f_[j], g[j], h[j], h[k]$	6	first solution
$q[d]\neq 0, r[d]=s[d]=0,\ f_[j], g[j], h[j]\neq 0$	5	second solution
$f_{\rm sd}[d],\ f_{\rm sd}[j], f_{\rm sd}[k], f_{\rm sd}[l]$	3	indeterminate

Table 4. A model similar to Georgi-Glashow with an additional coupling $h[k]$ in the off-diagonal and two additional couplings $f_{\rm sd}[k], f_{\rm sd}[l]$ in the semi-diagonal case.

13.1. **Off-diagonal cases: First solution with additional couplings.** We specify $f[5]=f_{ZW^15}f_{X^1Y^15}$ and $5\equiv W^2$ which leads to the other products

$$\begin{aligned}
g[6] &= f_{ZX^16}f_{W^1Y^16}\\
h[7] &= f_{ZY^17}f_{W^1X^17}\\
h[8] &= f_{ZY^18}f_{W^1X^18}\\
f[9] &= f_{ZX^19}f_{W^1Y^19}\\
g[9] &= f_{ZX^19}f_{W^1Y^19}\\
h[9] &= f_{ZY^19}f_{W^1X^19}.
\end{aligned}\tag{13.3}$$

Instead of $h[8]$ we could add an additional coupling $g[8]$ which leads in the end to the same solutions. We look now for solutions with this additional coupling in a theory with four masses.

There are four possibilities, corresponding to the masses $m[6], m[7]$ and $m[8]$; of this four, two lead to different solutions. In the first the masses m_6 and m_8 are set to $m[X]$, m_7 is set to $m[Y]$ or conversely $m[7]$ to $m[X]$ and $m[8]$ to $m[Y]$:

$$\begin{aligned}
m[1] &= m_Z^2\\
m[2] &= m[5] = m_W^2\\
m[3] &= m[6] = m[8] = m_X^2\\
m[4] &= m[7] = m_Y^2.
\end{aligned}\tag{13.4}$$

The corresponding solution is

$$\begin{aligned}
f[5] =&(m_X^2m_W^4m_Z^2 - m_W^4m_Y^2m_Z^2 + m_W^4m_Y^4 - m_X^2m_Y^2m_W^4 + m_Y^4m_W^2m_Z^2\\
&- m_X^2m_Y^2m_Z^2m_W^2 + 3m_X^4m_Y^2m_W^2 - m_W^2m_Y^6 - 2m_X^6m_W^2 + 2m_X^4m_Y^2m_Z^2\\
&- m_X^2m_Y^4m_Z^2 - m_X^6m_Z^2 - 2m_Y^4m_X^4 + 2m_X^8 - 2m_X^6m_Y^2 + 2m_Y^6m_X^2)m_W^2\Big/\\
&m_Z^2(m_W^6 - m_X^2m_W^4 - m_X^4m_W^2 + m_X^6 - m_Y^2m_W^4 + 2m_X^2m_Y^2m_W^2\\
&- m_X^4m_Y^2 - m_Y^4m_W^2 - m_X^2m_Y^4 + m_Y^6)m_X^2,
\end{aligned}\tag{13.5}$$

$$
\begin{aligned}
g[6] =& -(-2m_X^4 m_W^4 + m_W^4 m_Y^4 + m_X^2 m_W^4 m_Z^2 + m_X^2 m_Y^2 m_W^4 - m_W^4 m_Y^2 m_Z^2 \\
&+ 2m_X^6 m_W^2 + m_Y^4 m_W^2 m_Z^2 - m_X^4 m_Y^2 m_W^2 - m_X^2 m_Y^2 m_Z^2 m_W^2 - m_W^2 m_Y^6 \\
&+ 2m_X^4 m_Y^2 m_Z^2 - m_X^2 m_Y^4 m_Z^2 - m_X^6 m_Z^2) \Big/ \qquad (13.6) \\
&((m_W^6 - m_X^2 m_W^4 - m_X^4 m_W^2 + m_X^6 - m_Y^2 m_W^4 + 2m_X^2 m_Y^2 m_W^2 \\
&- m_X^4 m_Y^2 - m_Y^4 m_W^2 - m_X^2 m_Y^4 + m_Y^6) m_Z^2), \\
h[7] =& 1, \\
h[8] =& -m_Y^2(-3m_X^2 m_Y^4 m_W^2 + 2m_X^4 m_Y^2 m_W^2 - 2m_X^4 m_W^4 + m_X^6 m_W^2 + m_W^6 m_X^2 \\
&+ m_X^2 m_Y^2 m_W^4 + m_W^4 m_Y^4 - m_W^6 m_Y^2 + m_W^6 m_Z^2 + m_X^6 m_Z^2 - m_X^2 m_W^4 m_Z^2 \\
&+ m_X^2 m_Y^2 m_Z^2 m_W^2 - m_X^4 m_W^2 m_Z^2 - m_W^4 m_Y^2 m_Z^2 + m_X^2 m_Y^4 m_Z^2 - 2m_X^4 m_Y^2 m_Z^2) \Big/ \\
&m_Z^2 m_X^2 (m_W^6 - m_X^2 m_W^4 - m_X^4 m_W^2 + m_X^6 - m_Y^2 m_W^4 + 2m_X^2 m_Y^2 m_W^2 \\
&- m_X^4 m_Y^2 - m_Y^4 m_W^2 - m_X^2 m_Y^4 + m_Y^6), \\
f[9] =& 0, \\
g[9] =& 0, \\
h[9] =& 0.
\end{aligned}
\qquad (13.7)
$$

Most interesting in the solutions above is the question what is the order of the various products. If one has two breaking steps, as in $\mathfrak{su}(5)$, two corresponding gauge hierarchies are resulting. We check therefore what happens to the solutions if we assume

$$m_X^2, m_Y^2 \gg m_Z^2 \approx m_W^2$$

and in addition

$$m_Y^2 \simeq m_X^2 \pm \mathcal{O}(m_W^2) \qquad (13.8)$$

but we do not assume a specific mass relation. Here we note that the products obtained are not all of the same order. Instead we see

$$
\begin{aligned}
f[5] &= \mathcal{O}(1) \\
g[6] &= \mathcal{O}(\frac{m_Y^2}{m_W^2}) \\
h[7] &= \mathcal{O}(1) \\
h[8] &= \mathcal{O}(\frac{m_Y^2}{m_W^2}).
\end{aligned}
\qquad (13.9)
$$

One would expect $\mathcal{O}(1)$ for the products if a simple Lie algebra is realised. If the Lie algebra is semi-simple the orders can very well be different. We will dicuss this in more detail in section 17. The second solution reads with

$$
\begin{aligned}
m[1] &= m_Z^2 \\
m[2] &= m[5] = m_W^2 \\
m[3] &= m[8] = m_X^2 \\
m[4] &= m[6] = m[7] = m_Y^2 :
\end{aligned}
\qquad (13.10)
$$

$$\begin{aligned}
f[5] =&(-2m_Y^2m_X^4 + m_X^4m_Z^2 - m_X^2m_Z^4 + m_Y^2m_X^2m_W^2 - m_X^2m_W^2m_Z^2 \\
&+ m_X^2m_Y^2m_Z^2 - 2m_Y^4m_Z^2 - m_Y^4m_W^2 + 2m_Y^6 + m_W^2m_Z^4)m_W^4 \Big/ \\
&m_Y^2m_Z^2(2m_X^4m_W^2 - m_X^4m_Z^2 - 2m_W^4m_X^2 + m_Y^2m_X^2m_W^2 + m_X^2m_Y^2m_Z^2 \\
&+ m_W^4m_Z^2 - m_Y^2m_W^4 - m_Y^2m_W^2m_Z^2 + m_Y^4m_W^2), \\
g[6] =&1,
\end{aligned} \tag{13.11}$$

$$\begin{aligned}
h[7] =&(m_W^4m_Z^4 - m_W^4m_X^4 - m_W^4m_Y^4 + 2m_Y^2m_X^2m_W^4 - m_Y^2m_Z^4m_W^2 - m_Z^4m_X^2m_W^2 \\
&- m_X^4m_W^2m_Z^2 - m_Y^4m_Z^2m_W^2 + 2m_Y^2m_X^2m_W^2m_Z^2 - 2m_Y^4m_X^2m_W^2 + 2m_W^2m_X^6 \\
&+ 2m_W^2m_Y^6 - 2m_Y^2m_X^4m_W^2 + m_Y^4m_Z^4 + m_X^4m_Z^4 - 2m_Z^4m_Y^2m_X^2 + m_Y^2m_X^4m_Z^2 \\
&- m_Y^6m_Z^2 - m_X^6m_Z^2 + m_Y^4m_X^2m_Z^2)m_W^2 \Big/ \\
&(2m_W^4m_X^4 - m_X^2m_W^4m_Z^2 + m_Z^2m_W^4m_Y^2 + m_Y^2m_X^2m_W^2m_Z^2 + m_Y^4m_X^2m_Z^2 \\
&+ m_Y^2m_X^4m_W^2 + m_W^2m_Y^6 - 2m_Y^2m_X^4m_Z^2 - m_Y^4m_Z^2m_W^2 \\
&- 2m_W^2m_X^6 - m_Y^2m_X^2m_W^4 + m_X^6m_Z^2 - m_Y^4m_W^4)m_Z^2,
\end{aligned} \tag{13.12}$$

$$\begin{aligned}
h[8] =&(-2m_Y^4m_W^4 + m_W^6m_Z^2 - m_X^2m_W^4m_Z^2 - m_Z^2m_W^4m_Y^2 \\
&+ m_Y^2m_X^2m_W^4 - m_Y^4m_Z^2m_W^2 - 2m_Y^4m_X^2m_Z^2 + m_Y^2m_X^2m_W^2m_Z^2 \\
&+ m_W^6m_Y^2 + 2m_Y^4m_X^2m_W^2 + m_W^2m_Y^6 + m_Y^6m_Z^2 + m_Y^2m_X^4m_Z^2 \\
&+ m_W^4m_X^4 - m_W^6m_X^2 - 3m_Y^2m_X^4m_W^2)m_X^2 \Big/ \\
&m_Y^2(2m_W^4m_X^4 - m_X^2m_W^4m_Z^2 + m_Z^2m_W^4m_Y^2 + m_Y^2m_X^2m_W^2m_Z^2 \\
&+ m_Y^4m_X^2m_Z^2 + m_Y^2m_X^4m_W^2 + m_W^2m_Y^6 - 2m_Y^2m_X^4m_Z^2 \\
&- m_Y^4m_Z^2m_W^2 - 2m_W^2m_X^6 - m_Y^2m_X^2m_W^4 + m_X^6m_Z^2 - m_Y^4m_W^4), \\
f[9] =&0, \\
g[9] =&0, \\
h[9] =&0
\end{aligned} \tag{13.13}$$

and the orders are due to (13.8)

$$\begin{aligned}
f[5] &= \mathcal{O}(\frac{m_W^2}{m_Y^2}) \\
g[6] &= \mathcal{O}(1) \\
h[7] &= \mathcal{O}(\frac{m_W^2}{m_Y^2}) \\
h[8] &= \mathcal{O}(1).
\end{aligned} \tag{13.14}$$

We know that at least two of the new massive gauge bosons apart from the electroweak ones must have the same mass, otherwise they would not couple to the gluons and the photon at all. The following input makes therefore sense and one can even discuss more mass degeneracy in the theory. We specify $f[5] = f_{ZW^15}f_{X^1X^25}$ and $5 \equiv W^2$. This is our input:

$$\begin{aligned}
g[6] &= f_{ZX^16}f_{W^1X^26}\\
h[7] &= f_{ZX^27}f_{W^1X^17}\\
h[8] &= f_{ZX^28}f_{W^1X^18}\\
f[9] &= f_{ZX^19}f_{W^1X^29}\\
g[9] &= f_{ZX^19}f_{W^1X^29}\\
h[9] &= f_{ZX^29}f_{W^1X^19}
\end{aligned} \tag{13.15}$$

and the first solution now is with

$$\begin{aligned}
&m[1] = m_Z^2\\
&m[2] = m[5] = m_W^2\\
&m[3] = m[4] = m[6] = m[8] = m_X^2\\
&m[7] = m_Y^2 :
\end{aligned} \tag{13.16}$$

$$\begin{aligned}
f[5] = m_X^6 &+ m_Y^4 m_X^2 - 2m_X^4 m_Y^2 + m_X^2 m_Y^2 m_Z^2 - m_Z^2 m_Y^2 m_W^2 - m_Z^2 m_X^4\\
&+ m_Z^2 m_X^2 m_W^2 + m_X^2 m_W^2 m_Y^2 - m_X^4 m_W^2\\
&- m_X^4 m_Z^2 + m_X^2 m_Z^2 m_W^2 - m_X^4 m_W^2 \Big/\\
&m_Y^2 m_Z^2(-2m_X^2 + m_W^2),
\end{aligned} \tag{13.17}$$

$$\begin{aligned}
g[6] = m_X^2(&-m_X^4 m_W^2 - m_Y^2 m_Z^2 m_W^2 + m_X^2 m_Z^2 m_W^2 - m_X^2 m_Y^2 m_W^2 + 2m_W^2 m_Y^4\\
&- m_X^4 m_Z^2 + m_X^6 + m_Y^2 m_Z^2 m_X^2 + 2m_X^4 m_Y^2 - 3m_X^2 m_Y^4)\Big/\\
&m_Y^2 m_Z^2 m_W^2(-2m_X^2 + m_W^2),
\end{aligned} \tag{13.18}$$

$$\begin{aligned}
h[7] &= 1,\\
h[8] &= m_X^2(-2m_Z^2 m_X^2 m_W^2 + m_Z^2 m_W^4 + m_W^4 m_Y^2 - m_X^2 m_W^4 + 2m_X^4 m_W^2 - m_X^2 m_W^2 m_Y^2\\
&\quad - m_W^2 m_Y^4 - m_X^2 m_Y^2 m_Z^2 + m_Z^2 m_X^4 - m_X^6 - 2m_X^4 m_Y^2 + 3m_Y^4 m_X^2)\Big/\\
&\quad m_Z^2 m_Y^2 m_W^2(-2m_X^2 + m_W^2),\\
f[9] &= 0,\\
g[9] &= 0,\\
h[9] &= 0;
\end{aligned} \tag{13.19}$$

the products are of orders

$$\begin{aligned} f[5] &= \mathcal{O}(\frac{m_W^2}{m_Y^2}) \\ g[6] &= \mathcal{O}(\frac{m_Y^2}{m_W^2}) \\ h[7] &= \mathcal{O}(1) \\ h[8] &= \mathcal{O}(\frac{m_Y^2}{m_W^2}). \end{aligned} \tag{13.20}$$

For the second solution the mass degeneracy means:

$$\begin{aligned} m[1] &= m_Z^2 \\ m[2] &= m[5] = m_W^2 \\ m[3] &= m[4] = m[8] = m_X^2 \\ m[6] &= m[7] = m_Y^2 \end{aligned} \tag{13.21}$$

and we get

$$\begin{aligned} f[5] = {} & m_X^8 - 2m_Z^2 m_X^6 - 2m_X^6 m_W^2 + m_X^6 m_Y^2 + 4m_Z^2 m_X^4 m_W^2 - 5m_X^4 m_Y^4 + m_W^4 m_X^4 \\ & + m_X^4 m_Z^4 - 2m_W^4 m_X^2 m_Z^2 - m_W^4 m_X^2 m_Y^2 - 2m_X^2 m_W^2 m_Z^4 + 2m_Y^4 m_Z^2 m_X^2 \\ & + 3m_Y^6 m_X^2 - m_Y^2 m_Z^2 m_W^2 m_X^2 + 2m_Y^4 m_X^2 m_W^2 - m_Y^2 m_Z^4 m_X - 6m_Y^4 m_Z^2 m_W^2 \\ & + m_W^4 m_Z^2 m_Y^2 + m_Y^2 m_Z^2 m_W^2 + m_W^4 m_Z^4 \Big/ \\ & (m_X^4 - m_W^2 m_X^2 - m_X^2 m_Z^2 + 3m_Y^2 m_X^2 - 2m_W^2 m_Y^2 + m_W^2 m_Z^2) m_Y^2 m_Z^2, \\ g[6] = {} & 1, \end{aligned} \tag{13.22}$$

$$\begin{aligned} h[7] = {} & m_W^4 m_Z^2 - m_Y^4 m_W^2 - m_W^4 m_X^2 + m_W^4 m_Y^2 - m_Y^2 m_X^2 m_W^2 - m_X^2 m_Y^2 m_Z^2 \\ & - 2m_Z^2 m_W^2 m_X^2 + 3m_Y^4 m_X^2 - 2m_Y^2 m_X^4 + m_Z^2 m_X^4 + 2m_W^2 m_X^4 - m_X^6 \Big/ \\ & - m_W^2 m_X^4 - m_Y^2 m_X^2 m_W^2 + 2m_Y^4 m_W^2 - m_Y^2 m_Z^2 m_W^2 - m_Z^2 m_X^4 + m_X^2 m_Y^2 m_Z^2 \\ & + m_Z^2 m_W^2 m_X^2 + m_X^6 + 2m_Y^2 m_X^4 - 3m_Y^4 m_X^2, \end{aligned} \tag{13.23}$$

$$\begin{aligned} h[8] = {} & m_X^2(-2m_Y^4 m_Z^2 m_W^2 - 2m_Y^2 m_Z^2 m_W^2 m_X^2 + 2m_W^4 m_Z^2 m_Y^2 + m_W^4 m_Z^4 \\ & - 2m_W^4 m_X^2 m_Z^2 - 2m_X^2 m_W^2 m_Z^4 + 4m_Z^2 m_X^4 m_W^2 + m_W^4 m_Y^4 - m_Y^4 m_Z^4 \\ & + 2m_X^4 m_W^2 m_Y^2 - 2m_W^4 m_X^2 m_Y^2 - 2m_Y^6 m_W^2 - 2m_X^6 m_W^2 - 2m_Y^4 m_Z^2 m_X^2 \\ & + 8m_Y^6 m_X^2 - 6m_X^4 m_Y^4 + 2m_Y^4 m_X^2 m_W^2 + m_X^8 - 3m_Y^8 + m_W^4 m_X^4 + m_X^4 m_Z^4 \\ & + 4m_Y^6 m_Z^2 - 2m_Z^2 m_X^6) \Big/ \end{aligned} \tag{13.24}$$

$$
\begin{aligned}
& m_X^2 m_Z^2(-m_W^2 m_X^4 - m_Y^2 m_X^2 m_W^2 + 2m_Y^4 m_W^2 - m_Y^2 m_Z^2 m_W^2 - m_Z^2 m_X^4 \\
& \quad + m_X^2 m_Y^2 m_Z^2 + m_Z^2 m_W^2 m_X^2 + m_X^6 + 2m_Y^2 m_X^4 - 3m_Y^4 m_X^2), \\
f[9] &= 0, \\
g[9] &= 0, \\
h[9] &= 0
\end{aligned}
\tag{13.25}
$$

with orders

$$
\begin{aligned}
f[5] &= \mathcal{O}(\frac{m_W^2}{m_Y^2}) \\
g[6] &= \mathcal{O}(1) \\
h[7] &= \mathcal{O}(1) \\
h[8] &= \mathcal{O}(1).
\end{aligned}
\tag{13.26}
$$

REMARK: The masses m_7 and m_8 cannot be set equal here. Otherwise $h[7] = h[8]$ is only one parameter and the linear system has no solution. That is not only a negative result; it can either mean that all couplings of the form (13.3) are really zero or that one has to assume couplings of the form $f_{ZX^2W^{1,2}} f_{ZX^1W^{1,2}} \neq 0$. Then one finds solutions. We will discuss them in sect. 15. From the semi-diagonal case we find for this mass degeneracy that the sums of the form $\sum_i f_{X^1ZX^i} f_{ZX^3X^i}$ with $i = 1, \ldots, r+s$ must be zero, again under the assumption that there are no couplings from two weak bosons to one super-heavy.

13.2. **Off-diagonal cases: Second solution.** We specify indices 1 to 4 in order to obtain for the first product of structure constants $q[5] = f_{W^1W^25} f_{X^1X^25}$ and set $5 \equiv Z$, as before. This means for the other products of structure constants:

$$
\begin{aligned}
r[6] &= f_{W^1X^16} f_{W^2X^26} \\
s[7] &= f_{W^1X^27} f_{W^2X^17} \\
q[8] &= f_{W^1W^28} f_{X^1X^28} \\
r[8] &= f_{W^1X^18} f_{W^2X^28} \\
s[8] &= f_{W^1X^28} f_{W^2X^18}
\end{aligned}
\tag{13.27}
$$

where the index 8 denotes the (sum over) massless indices. Indices 5 and 6 are free. One can let them different from the particles $1, \ldots, 4$ or discuss mass degeneracies. In the following we choose $6 = Y$, $7 = X$:

$$
\begin{aligned}
q[5] =& -4m_Y^2 m_X^2 m_W^2 - m_Y^4 m_W^2 + 6m_X^2 m_Y^4 \\
& \quad - 4m_Y^2 m_X^4 - 4m_X^2 m_W^4 + 5m_X^2 m_W^2 - 2m_X^6 + m_W^6 \Big/ \\
& \quad (m_W^2 m_Z^2 m_Y^2), \\
r[6] =& -m_X^2(-3m_Y^4 + 2m_Y^2 m_X^2 + 2m_Y^2 m_W^2 + m_W^4 \\
& \quad - 2m_X^2 m_W^2 + m_X^4) \Big/ m_Y^2 m_W^4, \\
s[7] =& 1,
\end{aligned}
\tag{13.28}
$$

$$
\begin{aligned}
q[8] =&(m_W^8 - 4m_Y^2 m_X^2 m_W^4 - 4m_Y^2 m_X^4 m_W^2 + 2m_Z^2 m_X^4 m_Y^2 \\
&- 2m_Z^2 m_X^2 m_W^2 + m_Z^2 m_X^2 m_W^4 - 3m_Z^2 m_Y^4 m_X^2 \\
&+ 6m_Y^4 m_W^2 m_X^2 - m_W^4 m_Y^4 + 5m_W^4 m_X^4 \\
&+ 2m_Z^2 m_X^2 m_Y^2 m_W^2 + m_Z^2 m_Y^2 m_W^4 - 2m_X^6 m_W^2 \\
&+ m_Z^2 m_X^6 - 4m_W^6 m_X^2) \Big/ m_Z^2 m_Y^2 m_W^4, \\
r[8] =& 0, \\
s[8] =& 0.
\end{aligned} \tag{13.29}
$$

The same comes out with $6 = X$, $7 = Y$, there is only one solution. The orders are here

$$
\begin{aligned}
q[5] &= \mathcal{O}(\frac{m_Y^2}{m_W^2}) \\
r[6] &= \mathcal{O}(1) \\
s[7] &= \mathcal{O}(\frac{m_Y^2}{m_W^2}) \\
q[8] &= \mathcal{O}(\frac{m_Y^2}{m_W^2}).
\end{aligned} \tag{13.30}
$$

Setting $6 = X$, $7 = Y$ gives the same solution with $r[6]$ and $s[7]$ interchanged and starting with $q[5] = f_{W^1 W^2 5} f_{Y^1 Y^2 5}$ is just a change in m_X^2 and m_Y^2, the solution remains formally the same. For $m[6] = m[7] = m_Y^2$ we see

$$q[5] = -\frac{2m_X^2 m_Y^2 - 2m_Y^2 m_W^2 + m_Y^4 + m_W^4 + m_X^4 - 2m_W^2 m_X^2}{m_Y^2 m_X^2}, \tag{13.31}$$

$$r[6] = -1, \tag{13.32}$$

$$s[7] = 1, \tag{13.33}$$

$$q[8] = \frac{-2m_W^2 m_X^2 + m_X^4 + m_W^4 - 2m_X^2 m_Y^2 + 2m_Y^2 m_Z^2 + m_Y^4 - 2m_Y^2 m_W^2}{m_Z^2 m_X^2}. \tag{13.34}$$

All of these couplings are of order 1 and $r[8] = s[8] = 0$.

For $m_X^2 = m_Y^2$ we get here a solution that reads

$$q[5] := \frac{m_W^2(-4m_X^2 + m_W^2)}{m_X^2 m_Z^2}, \tag{13.35}$$

$$r[6] = -1, \tag{13.36}$$

$$s[7] = 1, \tag{13.37}$$

$$q[8] := -\frac{2m_X^2 m_Z^2 - 4m_X^2 m_W^2 + m_W^4}{m_X^2 m_Z^2} \tag{13.38}$$

and these couplings are again of order 1 with $m_X^2 \gg m_W^2 \approx m_Z^2$. This solution will be referred to in sect. 15. One has to allow for other couplings in the first off-diagonal and in the semi-diagonal solution in order to allow for a theory with 3 masses and the same order for all products.

14. Couplings to massless gauge bosons

We have analysed couplings from four fixed gauge particles to massive *and* massless indices. Therefore, most of the structure constants to massless gauge bosons do already appear in the massive choices. In case all or many massless particles are involved the relations tend to be trivially fulfilled, but one must be careful in the analysis since not all indices in the relations can become massless! It is not allowed to set some of the masses to zero in the solutions in appendix C. Since for products of the type $f_{\lambda\lambda\lambda} \cdot f_{\lambda\lambda\lambda}$ there are no restrictions cf. sect. 6, we have to look only on products $f_{\lambda\lambda\lambda} \cdot f_{XY\lambda}$. We go back to the systems of equations in sect. 11 and choose the indices 1 and 2 massless. Such an insertion is only permitted in (11.2), (11.3), (11.4), (11.5), (11.6), (11.9), (11.10), (11.12), (11.13), (11.16), (11.19). This smaller system has in the case of $f_{\lambda\lambda\lambda} \cdot f_{XY\lambda}$ with $m_X^2 \neq m_Y^2$ no solution which is quite obvious since we must have $f_{XY\lambda} = 0$ due to (2.90). More interesting is therefore the case $f_{\lambda_1\lambda_2 d} \cdot f_{XXd} := q'[9 = \lambda_i]$ which leads to

$$\begin{aligned} q'[5] &= 0 \quad \text{(index 5 must be massive)} \\ r'[6] &= f_{\lambda_1 X^1 6} f_{\lambda_2 X^2 6} \\ s'[7] &= f_{\lambda_1 X^2 7} f_{\lambda_2 X^1 7} \\ t'[8] &= f_{\lambda_1 X^2 8} f_{Z X^1 8} \\ q'[9] &= f_{\lambda_1 X^1 9} f_{\lambda_2 X^2 9} \\ r'[9] &= f_{\lambda_1 X^1 9} f_{\lambda_2 X^2 9} \\ s'[9] &= f_{\lambda_1 X^2 9} f_{\lambda_2 X^1 9} \end{aligned} \tag{14.1}$$

and in terms of the masses

$$r'[6] = 2\frac{m_X^2 m_6^2 (m_7^2 m_8^2 - m_X^2 m_7^2 - m_8^4 + m_X^2 m_8^2)}{(3m_6^2 m_7^2 - m_X^2 m_7^2 - m_6^2 m_X^2 - m_X^4) m_8^2 (m_6^2 - m_X^2)}, \tag{14.2}$$

$$s'[7] = \frac{-m_7^2(-4m_X^2 m_6^2 m_8^2 + m_X^4 m_6^2 + 3m_6^2 m_8^4 - m_X^2 m_8^4 + m_X^6)}{m_8^2(3m_6^2 m_7^4 - 4m_X^2 m_6^2 m_8^2 + m_X^4 m_6^2 - m_X^2 m_7^4 + m_X^6)}, \tag{14.3}$$

$$t'[8] = 1, \tag{14.4}$$

$$\begin{aligned} q'[9] = &- (3m_6^4 m_8^2 m_7^4 - 3m_6^4 m_7^2 m_8^4 - m_X^4 m_6^4 m_7^2 + m_X^4 m_6^4 m_8^2 - 2m_X^2 m_6^2 m_8^2 m_7^4 \\ &- 2m_X^4 m_6^2 m_7^4 + 2m_X^6 m_6^2 m_7^2 + 2m_X^2 m_6^2 m_7^2 m_8^4 + 2m_X^4 m_6^2 m_8^4 - 2m_X^6 m_6^2 m_8^2 \\ &+ m_8^2 m_X^4 m_7^4 + m_7^2 m_X^8 - m_7^2 m_X^4 m_8^4 - m_8^2 m_X^8) \Big/ \\ &(m_8^2 (m_6^2 - m_X^2)(3m_6^2 m_7^2 - m_X^2 m_7^2 - m_6^2 m_X^2 - m_X^4)(m_7^2 - m_X^2)), \end{aligned} \tag{14.5}$$

$$r'[9] = s'[9] = 0. \tag{14.6}$$

With $m[6] = m[X]$, $m[7] = m[X]$ and $m[8] = m[Y]$ we arrive at

$$r'[6] = \sum_j f_{\lambda_1 X^1 X^j} f_{\lambda_2 X^2 X^j} = s'[7] = \sum_k f_{\lambda_1 X^2 X^k} f_{\lambda_2 X^1 X^k}. \tag{14.7}$$

All other products are zero, in particular $r'[6]$:

$$r'[6] = -4m_X^6 + m_X^6 + 3m_X^6 - m_X^6 + m_X^6 = 0,$$

hence it is not possible to derive pure mass relations from the fact that $r'[6]$ must be zero, as argued in sect. 6. With $m[6] = m[Y]$, $m[7] = m[X]$ and $m[8] = m[Y]$ we are left with the restriction

$$s'[7] = f_{\lambda_1 X^2 X^j} f_{\lambda_2 X^1 X^j} = q'[9] = f_{\lambda_1 X^1 9} f_{\lambda_2 X^2 9} = \sum_d f_{\lambda_1 X^1 d} f_{\lambda_2 X^2 d} \tag{14.8}$$

which must be zero. Therefore, such a mass degeneracy is not possible. Eq. (14.7) is the only meaningful solution; we will refer to it later, when necessary. There are no restrictions from the semi-diagonal restrictions here, because couplings of the form $f_{\lambda X \lambda}$ must vanish.

15. Non-standard weak couplings

So far we have considered theories with the normal weak coupling $f_{W^1 W^2 Z}$. We did not allow for other couplings to the W^1 *and* W^2, Z *and* W^1 or Z *and* W^2. In the Georgi-Glashow model this makes sense, since the Z^0 is the only neutral gauge boson and the new gauge particles have charges $Q_{X,Y} \neq \pm 1$. In the general case though such couplings are no more forbidden. As a matter of course, one would expect them to be of order $\mathcal{O}(\frac{m_W^2}{m_Y^2})$, so that they cannot be seen at the electroweak scale. In case one allows for such "non-standard" couplings, the situation is much more involved because in the linear systems we can in principle then add couplings at every place. A general treatment is therefore not possible; in this section we focus on smaller theories with mass degeneracies.

The procedure is similar as before: The diagonal rank 6 and rank 5 systems are first checked. Then we look particularly on the semi-diagonal case.

15.1. A theory with additional charged and neutral gauge bosons. Three masses. [20]

We consider here the case of three masses, i.e. the two masses of the electroweak bosons m_W, m_Z and one mass m_X for all super-heavy particles with $m_X^2 \gg m_W^2, m_Z^2$; we look directly for a theory with charged and neutral X bosons. One quickly sees that a theory with only neutral X particles is not tenable: In such a case couplings of the form f_{ZWX} would have to vanish. From our first solution in the off-diagonal case that is of rank 6 (rank 3 without couplings to massless gauge bosons) we would be forced to have non-vanishing structure constants of the form $f_{ZXW^2} f_{W^1 X W^2}$, but they are zero.

If all new X particles where equally charged, couplings of the form f_{XXX} must be zero. Again, the first solution necessitates non-vanishing couplings of the form f_{WXW} with a (some) neutral X-boson(s).

We consider two special cases:

Case $f_{XXX} = 0$: In this case we can derive from the first off-diagonal solution (13.15) with $m[5] = m_W^2$, $m[6] = m_X^2 = m[8]$ and $m[7] = m_W^2$ expressions for the

[20] In [26] it was shown that a purely massive theory with *three gauge bosons* leads to a total mass degeneracy $m_1 = m_2 = m_3$, otherwise there is no solution. Our treatment here is based on any number of gauge bosons with three common masses.

couplings

$$\begin{aligned}
f[5=W] &= f_{ZW^1W^2} f_{X^1X^2W^2}\\
g[6=X] &= \sum_i f_{ZX^1X^i} f_{W^1X^2X^i}\\
h[7=W] &= f_{ZX^2W^2} f_{W^1X^1W^2}\\
h[8=X] &= \sum_i f_{ZX^2X^i} f_{W^1X^1X^i}\\
f[9=\lambda_i,\gamma] &= f_{ZX^19} f_{W^1X^29}\\
g[9] &= f_{ZX^19} f_{W^1X^29}\\
h[9] &= f_{ZX^29} f_{W^1X^19}
\end{aligned} \tag{15.1}$$

under the assumption that $g[W]=0$ the expressions

$$f[5]=1, \tag{15.2}$$

$$g[6]=\frac{-m_X^2m_W^2(-2m_X^2m_W^2+2m_W^4-m_Z^2m_W^2+m_Z^2m_X^2-m_X^4)}{m_Z^2m_W^2-m_Z^2m_X^2-2m_X^2m_W^2+m_X^4}, \tag{15.3}$$

$$h[7]=\frac{-m_W^2m_Z^2(-2m_X^2+m_W^2)}{(m_Z^2m_W^2-m_Z^2m_X^2-2m_X^2m_W^2+m_X^4)(m_W^2-m_X^2)}, \tag{15.4}$$

$$h[8]=\frac{(m_Z^2m_W^4+m_X^2m_W^4-3m_W^2m_Z^2m_X^2+m_X^4m_Z^2-m_X^6)m_X^2}{(m_Z^2m_W^2-m_Z^2m_X^2-2m_X^2m_W^2+m_X^4)m_W^2(m_W^2-m_X^2)}, \tag{15.5}$$

$$g[9]=h[9]=0 \tag{15.6}$$

for the structure constants hold true and the products are of orders

$$\begin{aligned}
f[5] &= \mathcal{O}(1)\\
g[6] &= \mathcal{O}(\frac{m_X^2}{m_W^2})\\
h[7] &= \mathcal{O}(\frac{m_W^4}{m_X^4})\\
h[8] &= \mathcal{O}(\frac{m_X^2}{m_W^2}).
\end{aligned} \tag{15.7}$$

If $g[W]\neq 0$ but $g[X]=0$ an other solution comes out with a coupling to W instead of one to X in g: $m[6]=m_W^2$ and reads

$$\begin{aligned}
f[5]=&(-m_W^6m_Z^2m_X^2-2m_W^6m_Z^4+5m_Z^4m_W^4m_X^2-4m_Z^4m_X^4m_W^2\\
&+6m_Z^2m_W^2m_X^6+m_X^{10}-5m_Z^2m_X^4m_W^4-4m_X^2m_W^8+2m_Z^2m_W^8-2m_X^8m_W^2\\
&-2m_Z^2m_X^8-3m_X^6m_W^4+m_Z^4m_X^6+8m_X^4m_W^6)\Big/m_W^2m_Z^2(-3m_W^2m_Z^2m_X^2\\
&+m_X^2m_W^4+m_X^4m_Z^2-m_X^6+m_W^4m_Z^2),
\end{aligned} \tag{15.8}$$

$$g[6]=\frac{(-4m_X^2m_W^4+m_X^4m_W^2+2m_W^2m_Z^2m_X^2+m_X^6+2m_W^6-m_X^4m_Z^2-m_W^4m_Z^2)}{(-3m_W^2m_Z^2m_X^2+m_X^2m_W^4+m_X^4m_Z^2-m_X^6+m_W^4m_Z^2)}, \tag{15.9}$$

$$h[7] = m_X^2(-4m_Z^2 m_W^6 + 2m_W^2 m_Z^4 m_X^2 - 4m_Z^2 m_W^2 m_X^4 - m_X^8 + 6m_Z^2 m_W^4 m_X^2 + 4m_W^8 + 2m_Z^2 m_X^6 + 3m_X^4 m_W^4 - m_X^4 m_Z^4 - 8m_X^2 m_W^6 + 2m_X^6 m_W^2) \Big/ m_W^2 m_Z^2(-3m_W^2 m_Z^2 m_X^2 + m_X^2 m_W^4 + m_X^4 m_Z^2 - m_X^6 + m_W^4 m_Z^2), \tag{15.10}$$

$$h[8] = 1 \tag{15.11}$$

with the orders

$$\begin{aligned} f[5] &= \mathcal{O}(\frac{m_X^4}{m_W^4}) \\ g[6] &= \mathcal{O}(1) \\ h[7] &= \mathcal{O}(\frac{m_X^4}{m_W^4}) \\ h[8] &= \mathcal{O}(1). \end{aligned} \tag{15.12}$$

For both $g[W] \neq g[X] \neq 0$ our rank 6 system is not sufficient to determine all the couplings.

From the second solution in appendix C.2 we have already found a solution in the case of three masses m_Z^2, m_W^2 and m_X^2: (13.35) - (13.38). But this is no longer the only one. We find other solutions for the couplings (13.27) by allowing for couplings of the form f_{WXZ}. With $f_{XXX} = 0$ and $m[5] = m_Z^2$, $m[6] = m_Z^2$, $m[7] = m_X^2$:

$$\begin{aligned} q[5 = Z] &= f_{W^1 W^2 Z} f_{X^1 X^2 Z} \\ r[6 = Z] &= f_{W^1 X^1 Z} f_{W^2 X^2 Z} \\ s[7 = X] &= \sum_i f_{W^1 X^2 X^i} f_{W^2 X^1 X^i} \\ q[8 = \gamma, \lambda_i] &= f_{W^1 W^2 8} f_{X^1 X^2 8} \\ r[8] &= f_{W^1 X^1 8} f_{W^2 X^2 8} \\ s[8] &= f_{W^1 X^2 8} f_{W^2 X^1 8} \end{aligned} \tag{15.13}$$

we find

$$q[5] := (-m_W^2 m_Z^4 - 4m_X^2 m_W^4 + m_W^6 + 5m_X^4 m_W^2 - 4m_X^2 m_Z^2 m_W^2 - 4m_X^4 m_Z^2 - 2m_X^6 + 6m_X^2 m_Z^4) \Big/ m_W^2 m_Z^4, \tag{15.14}$$

$$r[6] = 1, \tag{15.15}$$

$$s[7] = \frac{m_X^2(m_W^4 - 2m_X^2 m_W^2 + 2m_W^2 m_Z^2 + 2m_X^2 m_Z^2 + m_X^4 - 3m_Z^4)}{m_X^2 m_Z^4}, \tag{15.16}$$

$$q[8] = (-3m_Z^2 m_X^2 m_W^4 + 5m_X^4 m_W^4 - 4m_X m_W^6 + m_W^8 - 6m_Z^2 m_X^4 m_W^2 + 8m_X^2 m_W^2 m_Z^4 + 2m_X^4 m_Z^4 + m_Z^2 m_X^6 - 3m_X^2 m_Z^6 - 2m_X^6 m_W^2) \Big/ m_W^4 m_Z^4. \tag{15.17}$$

Except for $r[6]$, all products are of order $\mathcal{O}(\frac{m_X^6}{m_W^6})$. The same solution comes out, when setting $m[5] = m_Z^2$, $m[6] = m_X^2$, $m[7] = m_Z^2$. The last possibility $m[5] = m_Z^2$,

$m[6] = m_Z^2$, $m[7] = m_Z^2$ yields

$$q[5] := \frac{-2m_X^2 m_Z^2 - 2m_W^2 m_Z^2 + m_Z^4 + m_W^4 + m_X^4 - 2m_W^2 m_X^2}{m_Z^4}, \tag{15.18}$$

$$r[6] = -1, \tag{15.19}$$

$$s[7] = 1, \tag{15.20}$$

$$q[8] = -\frac{m_W^4 + 3m_Z^4 - 2m_W^2 m_X^2 - 2m_W^2 m_Z^2 - 2m_X^2 m_Z^2 + m_X^4}{m_Z^4} \tag{15.21}$$

with

$$q[5] = q[8] = \mathcal{O}(\frac{m_X^4}{m_Z^4}). \tag{15.22}$$

Finally from the semi-diagonal restrictions we find in addition:

$$\sum_j f_{ZX^1X^j} f_{ZX^2X^j} = 0; \; f_{ZX^1W^1} f_{ZX^2W^1} + f_{ZX^1W^2} f_{ZX^2W^2} = 0 \tag{15.23}$$

which necessitates a selective coupling structure for this model. Finally from the couplings to massless gauge bosons we are left with (14.7). This restriction remains valid in the second case.

Case $f_{XXX} \neq 0$: In this case we have two couplings in f, one in g and one in h for the first solution. There are three solutions here; the first with the couplings

$$\begin{aligned}
f[5 = W] &= f_{ZW^1W^2} f_{X^1X^2W^2} \\
f[6 = X] &= \sum_i f_{ZW^1X^i} f_{X^1X^2X^i} \\
g[7 = X] &= \sum_i f_{ZX^1X^i} f_{W^1X^2X^i} \\
h[8 = W] &= f_{ZX^2W^2} f_{W^1X^1W^2} \\
f[9 = \lambda_i, \gamma] &= f_{ZX^19} f_{W^1X^29} \\
g[9] &= f_{ZX^19} f_{W^1X^29} \\
h[9] &= f_{ZX^29} f_{W^1X^19}
\end{aligned} \tag{15.24}$$

that reads

$$\begin{aligned}
f[5] =& 3m_X^4 m_W^4 - 2m_W^6 m_X^2 + m_Z^2 m_W^6 - 3m_Z^2 m_W^2 m_X^4 - m_Z^4 m_W^4 + 3m_Z^4 m_W^2 m_X^2 \\
& - m_Z^4 m_X^4 + 2m_Z^2 m_X^6 - m_X^8 \Big/ (2m_W^6 - 3m_W^4 m_X^2 - 2m_W^4 m_Z^2 \\
& + 3m_W^2 m_Z^2 m_X^2 - m_X^4 m_Z^2 + m_X^6) m_X^2,
\end{aligned} \tag{15.25}$$

$$f[6] = \frac{(m_W^4 m_Z^2 + m_W^4 m_X^2 - 3m_W^2 m_Z^2 m_X^2 + m_X^4 m_Z^2 - m_X^6) m_Z^2}{m_X^2 (2m_W^6 - 3m_W^4 m_X^2 - 2m_W^4 m_Z^2 + 3m_W^2 m_Z^2 m_X^2 - m_X^4 m_Z^2 + m_X^6)}, \tag{15.26}$$

$$g[7] = \frac{m_W^4 m_Z^2}{(2m_W^4 - m_X^2 m_W^2 - 2m_Z^2 m_W^2 + m_Z^2 m_X^2 - m_X^4) m_X^2}, \tag{15.27}$$

$$h[8] = 1, \; f[9] = g[9] = h[9] = 0. \tag{15.28}$$

The same solution results if one sets $m[7] = m_W^2$ and $m[8] = m_X^2$. The second solution comes with $m[7] = m[8] = m_W^2$:

$$f[5] = -\frac{(m_W^2 + m_Z^2 - m_X^2)m_X^2}{(m_W^2 - m_X^2)m_W^2}, \tag{15.29}$$

$$f[6] := \frac{-3m_X^2 m_W^2 - m_X^2 m_Z^2 + m_X^4 + 2m_W^4}{(m_W^2 - m_X^2)m_W^2}, \tag{15.30}$$

$$g[7] = 1 = -h[8];\ f[9] = g[9] = h[9] = 0. \tag{15.31}$$

The third with $m[7] = m[8] = m_X^2$ is:

$$f[5] := \frac{2m_X^4 - 2m_X^2 m_W^2 - 2m_Z^2 m_X^2 + m_W^2 m_Z^2}{(m_X^2 - m_W^2)m_X^2}, \tag{15.32}$$

$$f[6] := \frac{m_Z^2(2m_X^2 - m_W^2)}{m_X^2(m_X^2 - m_W^2)}, \tag{15.33}$$

$$g[7] = 1 = -h[8];\ f[9] = g[9] = h[9] = 0. \tag{15.34}$$

From the second off-diagonal solution we have, splitting the couplings q

$$\begin{aligned}
q[5 = Z] &= f_{W^1W^2Z} f_{X^1X^2Z} \\
q[6 = X] &= \sum_i f_{W^1W^2X^i} f_{X^1X^2X^i} \\
r[6] &= 0 \\
s[7 = X] &= f_{W^1X^2X} f_{W^2X^1X} \\
q[8 = \gamma, \lambda_i] &= f_{W^1W^28} f_{X^1X^28} \\
r[8] &= f_{W^1X^18} f_{W^2X^28} \\
s[8] &= f_{W^1X^28} f_{W^2X^18}
\end{aligned} \tag{15.35}$$

the solution

$$q[5] = \frac{m_X^2}{m_Z^2}, \tag{15.36}$$

$$q[6] = 1, \tag{15.37}$$

$$s[7] = 0, \tag{15.38}$$

$$q[8] = \frac{m_X^2 - m_Z^2}{m_Z^2}, \tag{15.39}$$

$$r[8] = s[8] = 0. \tag{15.40}$$

The same solution comes out assuming $s[7] = 0$. Setting $m[7] = m_W^2$ or $m[8] = m_W^2$ is not possible due to antisymmetry.

One could also look for solutions with $f_{ZWW} f_{XXW} = 0$ but we abstain from this possibility because solutions without the electroweak coupling are uninteresting. Already in the cases considered there remains the possibility to have more couplings than we can determine with a rank 6 system (or 5 respectively).

We see that both of the off-diagonal solutions yield products of structure constants of different orders if “non-standard” couplings are involved. This does not exclude a model with three masses but it is not likely to be realised based on a simple Lie algebra.

15.2. **A theory with additional charged and neutral gauge bosons. Four masses.** Motivated by (13.35) - (13.38) we have studied a theory with three masses and found no solution with structure constants of the same order. In this section we start from (13.31) - (13.34). This was the only solution with four masses yielding products of structure constants of the same order. We will also here add more couplings and look for solutions of order one. Yet there are too many possible combinations in the products; we can only look on some special cases.

In the derivation of (13.31) we put $m[6] = m[7] = m_Y^2$ and have therefore in r and s only couplings of the form $f_{WXY}f_{WXY}$, leading to a solution with products of structure constants of the same order of magnitude. The question is now whether in the *first* off-diagonal solution there is a possibility for order one structure constants when allowing for couplings of the form f_{ZXW} in g and in h:

$$\begin{aligned}
f[5=W] &= f_{ZW^1W^2}f_{X^1Y^1W^2} \\
g[6=Y] &= \sum_j f_{ZX^1Y^j}f_{W^1Y^1Y^j} \\
h[7=W] &= f_{ZY^1W^2}f_{W^1X^1W^2} \\
h[8=Y] &= \sum_j f_{ZY^1Y^j}f_{W^1X^1Y^j} \\
f[9] &= f_{ZX^19}f_{W^1Y^19} \\
g[9] &= f_{ZX^19}f_{W^1Y^19} \\
h[9] &= f_{ZY^19}f_{W^1X^19}.
\end{aligned} \tag{15.41}$$

Under the gauge hierarchy (13.8) not all of these products come out of the same order. With $g[6 = W]$, $h[7 = Y]$ and $h[8 = W]$ we are led to the same solution. Now we replace Y by X in g and h:

$$\begin{aligned}
g[6=X] &= \sum_i f_{ZX^1X^i}f_{W^1Y^1X^i} \\
h[7=W] &= f_{ZY^1W^2}f_{W^1X^1W^2} \\
h[8=X] &= \sum_i f_{ZY^1X^i}f_{W^1X^1X^i}.
\end{aligned} \tag{15.42}$$

Also here the products are not of equal order. What remains is to look for both X- and Y-couplings in the products g and h. There are six cases to check here. Since non of them contains a solution with all products of order one, one must conclude that either there are more couplings that cannot be determined by our rank 6 system or that the breaking scheme, the gauge hierarchy assumed, is too simple.

16. Jacobi-Identity — a consistency check

We know that in second order the Jacobi-identity (2.75)

$$f_{abc}f_{cef} + f_{eac}f_{cbf} + f_{ebc}f_{acf} = 0 \tag{16.1}$$

is fulfilled. In our solutions from the rank 3 equation system e.g. we have restrictions to 4 sums of products of structure constants. We cannot know whether these are already all couplings. The sum over c though goes over all couplings. Therefore, a check of the Jacobi-identity is more than a consistency check: if the identity is not already fulfilled we see, that more couplings to a gauge boson with a different mass are necessary.

In the semi-diagonal case when two indices are equal, the Jacobi identity is always fulfilled. So we must only look at the off-diagonal case. We check the Jacobi-identity for the second solution: $a = W^1$, $b = W^2$, $e = X^1$, $f = X^2$, if we arrange the indices according to table 1, we get

$$f_{W^1W^2d}f_{X^1X^2d} + f_{W^1W^2Z}f_{X^1X^2Z} + f_{X^1jW^1}f_{X^2jW^2} + f_{X^1jW^2}f_{X^2jW^1} = 0 \quad (16.2)$$

or in our notation:

$$q[5] + q[8] - r[6] + s[7] = 0 \quad (16.3)$$

which is true. We checked this also for the general solutions in appendix C.

17. Gauge Hierarchies

Quite soon after the proposals of grand unification it was noted that the specific gauge hierarchy, experimentally implied through proton decay, can only be achieved by fine-tuning the Higgs potential (in every order); the gauge hierarchy is therfore not natural [36], sect. 6.9. In [46] it is furthermore shown that the unification scale in $SU(5)$ must be below the Planck scale, so one has in the end several breaking stages. In the meantime many other possible gauge hierarchies have been advocated, there is no general agreement on a specific one.

In this work, sect. 15, we have not assumed a special mass hierarchy at the beginning. Do we know whether the gauge hierarchy comes out by gauge invariance? For answering this question, let us go back to (2.89) with $a = X^1$, $b = X^2$ and $c = X^3$ with degenerate mass m_X and assuming that the only massive particle coupling to two X is the Z (as it is the case for $SU(5)$). Then we have

$$0 = -\sum_d f_{X^1X^2d}f_{X^2X^3d} + \frac{3}{4}\frac{m_Z^2}{m_X^2} - 1$$

or

$$\frac{\sum_d f_{X^1X^2d}f_{X^2X^3d}}{f_{X^1X^2Z}f_{X^2X^3Z}} + 1 = \frac{3}{4}\frac{m_Z^2}{m_X^2}; \quad (17.1)$$

that means

$$\frac{\sum_d f_{X^1X^2d}f_{X^2X^3d}}{f_{X^1X^2Z}f_{X^2X^3Z}} \approx -1. \quad (17.2)$$

Such a relation can only be realised by an (infinitesimal) rotation of the structure constants, it cannot directly be expected from values with structure constants of Lie type. When we abstain from concrete values for the structure constants (as we did in sect. 9), we expect that the gauge hierarchy is coming out from gauge invariance, since one can express products of structure constants by the masses of the corresponding gauge bosons only. We did therefore not *assume* a gauge hierarchy at the beginning; we only checked in the end whether the concrete $SU(5)$ gauge hierarchy results. This is not the case. With our arguments we make thus evident that $SU(5)$ is not quantum gauge invariant.

This is also relevant in $SO(10)$ grand unification, because it implies that the breaking chain $SO(10) \longrightarrow SU(5) \times U(1) \longrightarrow SU(5) \longrightarrow SU(3) \times SU(2) \times U(1) \longrightarrow SU(3) \times U(1)$ cannot be realised.[21] The gauge hierarchy must be different: There is the second chain $SO(10) \longrightarrow SU(4) \times SU(2) \times SU(2) \longrightarrow SU(3) \times SU(2) \times U(1) \longrightarrow SU(3) \times U(1)$. This breaking scheme has been proposed by Georgi and independently

[21] Experimentally this seems to be clear because in $SU(5) \times U(1)$ proton lifetime is even shorter than in ordinary $SU(5)$, [11], § 7.4.

by Fritzsch and Minkowski [33]. The unification group $SU(4) \times SU(2) \times SU(2)$ was considered by Pati and Salam in 1973 [32], leading to a right-left symmetric unification model.[22] In such a breaking scheme the solutions presented in appendix C could lead to interesting results. It might though be that they do not determine the theory entirely, when the ranks of our systems of equations do not increase in a more general setting, i.e. when allowing for even more non-vanishing couplings.

One can also not be sure whether the "desert" between the breaking stages is really empty [47]. If there are intermediate stages between the electroweak and the unification scale, simple (MSSM) $SU(5)$ and similar models must be discarded. We do not know whether, apart from gravitation, the three couplings, strong, weak and electromagnetic are really exhaustive, leading to a simple gauge hierarchy.

18. Conclusions

We have analysed restrictions by second order quantum gauge invariance in the case of grand unification. What are the results? In a first step (sect. 8) we have checked $SU(5)$ with concrete structure constants, finding contradictions. The coupling structure seems to be too selective. Also a simple rotation does not help: Several such special rotations can be accomplished but without success; considering a general rotation leads to too many free parameters. Because of this fact, we have left the coupling structure as open as possible in the following. We have only restricted couplings according to charge conservation and gauge invariance (sect. 9). With this coupling structure we arrive at a small system of equations that does not allow for non-trivial solutions again. The coupling structure must be even less restrictive. Therefore we have permitted a fairly open coupling structure, arriving at a far bigger system of equations, that is linear in the products of structure constants. With one product more than in the Georgi-Glashow model we find solutions (appendix C). These solutions we have analysed in the case of a theory with three and four masses with standard couplings only (sect. 13), finding no solution with all products of order one. This would be desirable for a gauge theory with a simple gauge group. Also allowing for non-standard couplings (sect. 15) does not help. We thus come in several steps to the conclusion that the specific $SU(5)$ breaking pattern $SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$ is not mirrored in our restrictions from quantum gauge invariance.

Because of this disturbing result, we carefully checked all our computations. The relations leading to a contradiction come out independently from different sectors (sect. 2.3). In the cases considered, each sector is of equal strength, giving us an excellent consistency check. Together with A. Schnyder [26] we controlled the gauge conditions again. We therefore think that an error in our basics is unlikely. The computations on the computer we have carried out primarily on Maple 7.0 and used Mathematica 5.0 to check the results.

Our results are not only negative — on the contrary. In view of all restrictions (sect. 11) it is amazing that the resulting systems of equations are all of low rank. A full determination of the coupling structure in (grand) unification by gauge invariance alone is thus improbable. Although the breaking scheme $SO(10) \longrightarrow SU(5) \times U(1) \longrightarrow$ GWS is not possible from our point of view, there remain possibilities as $SO(10) \longrightarrow SU(4) \times SU(2) \times SU(2) \longrightarrow$ GWS, containing the famous

[22] There is a possibility of breaking $SO(10) \longrightarrow SU(2) \times U(1)_R \times SU(4)$ as well [34].

Pati-Salam group. If the ranks of our linear sets of equations still do not increase, such a model is likely to be realised.

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Part 3. Appendices

Appendix A. Basis for (classical) Lie algebras

The structure constants in our second order conditions for gauge invariance are, as usual, totally antisymmetric and fulfil the Jacobi identity. Further we know that the fields are Hermitian. This knowledge in mind, we investigate Lie algebras and construct, step by step, the desired structure constants. In the literature we rarely find explicit lists of generators and structure constants beyond $\mathfrak{su}(3)$. For our primary case $\mathfrak{su}(5)$, there does exist a list, compiled by Hayashi et al. in [29]. Yet in order to understand how this comes about and to check them, we are going through the construction from a purely mathematical viewpoint, leading to the so called Cartan Weyl basis. It is the "via regia" for semisimple Lie algebras. For the mathematical background we work with a recent book on that topic written by J. Fuchs and Ch. Schweigert [3].

Let us start with some general remarks: The most important Lie algebras $\mathfrak{g}$ for applications in physics are the Abelian and simple Lie algebras and their direct sums [For an *Abelian* Lie algebra $[\mathfrak{g},\mathfrak{g}] = 0$ holds and a *simple* Lie algebra is one with no proper ideals and which is not Abelian. A direct sum of simple Lie algebras is called *semisimple* and a direct sum of simple and Abelian Lie algebras is referred to as *reductive*].

In the construction of Cartan Weyl bases we first identify a certain Abelian subalgebra of $\mathfrak{g}$. If the underlying field is algebraically closed, as is $\mathbb{C}$, in any semisimple Lie algebra we find elements which commute with each other. One chooses a maximal set of linearly independent elements and denotes them as H^i with $i = 1, \dots, r$. The linear hull spanned by these elements form the *Cartan subalgebra* $\mathfrak{g}_\circ$ of $\mathfrak{g}$. The Cartan subalgebra is unique only up to automorphisms but this does not lead to arbitrariness in the description; all Cartan subalgebras have dimension r which is called the *rank* of $\mathfrak{g}$.

All elements H of the Cartan subalgebra $\mathfrak{g}_\circ$ commute with each other. This means that they are simultaneously diagonalisable. The dual space of the Cartan subalgebra is spanned by some elements E in the sense that

$$[H, E] =: \mathrm{ad}_H(E) = \alpha_E(H)E$$

where ad is the adjoint map, as just defined. For any fixed element $E \in \mathfrak{g}$ the eigenvalue $\alpha_E(H)$ of E is some complex number which depends linearly on $H \in \mathfrak{g}_\circ$. Therefore $\alpha_E(H)$ is a linear function $\alpha_E : \mathfrak{g}_\circ \longrightarrow \mathbb{C}$, i.e. an element of a vector space $\mathfrak{g}_\circ^\star$ dual to $\mathfrak{g}_\circ$. The eigenvalues of ad_H are the *roots* of the characteristic equation for H. We denote the elements of the basis $\mathcal{B}$ of $\mathfrak{g}$ apart from the H_i's as E^α in the sense that for each α the eigenvalue $\alpha^i := \alpha(H^i)$ is non-vanishing for at least one value of i.

For $\mathfrak{g}$ we have then the following so called *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_\circ \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$$

of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{g}_0$. The set of all roots $\mathfrak{g}_\alpha$ is spanned by non-degenerate elements E^α, i.e. all roots are one dimensional [for a proof cf. [3],

TABLE 5. Roots for $\mathfrak{A}_4(\mathfrak{sl}(5,\mathbb{C}))$. The e_i are basis vectors.

Simple positive roots	Non-simple positive roots	Negative roots	Non-simple negative roots
$\alpha^1 := e_1 - e_2$	$\alpha^5 := e_1 - e_3$	$\alpha^{17} := e_5 - e_4$	$\alpha^{11} := e_5 - e_3$
$\alpha^2 := e_2 - e_3$	$\alpha^6 := e_1 - e_4$	$\alpha^{18} := e_4 - e_3$	$\alpha^{12} := e_5 - e_2$
$\alpha^3 := e_3 - e_4$	$\alpha^7 := e_1 - e_5$	$\alpha^{19} := e_3 - e_2$	$\alpha^{13} := e_5 - e_1$
$\alpha^4 := e_4 - e_5$	$\alpha^8 := e_2 - e_4$	$\alpha^{20} := e_2 - e_1$	$\alpha^{14} := e_4 - e_2$
	$\alpha^9 := e_2 - e_5$		$\alpha^{15} := e_4 - e_1$
	$\alpha^{10} := e_3 - e_5$		$\alpha^{16} := e_3 - e_1$

p. 88]. One further shows that the roots span all of $\mathfrak{g}_0^\star = \mathfrak{g} \setminus \mathfrak{g}_0$ and that, given a root α, its only multiples are $\pm\alpha$.

Taking the H's and the E's we get the Cartan Weyl basis

$$\mathcal{B} = \{H^i | i = 1, \dots, r\} \cup \{E^\alpha\}.$$

The generators E^α are called *step* or *ladder* operators. Note that they are not hermitian operators. This will be desirable for physical generators.

A.1. **The Cartan Catalogue of simple Lie algebras.** Let us briefly summarise the catalogue of simple Lie algebras as originally obtained by Cartan in his thesis [6] (quoted after [5], ch. 15, Appendix B) in 1894. When considering grand unification schemes it is indispensable to know all of these. The subscript n is the rank of the Lie algebra. We indicate here which groups rely on these algebras.

- $\mathfrak{A}_n$: $SU(n+1)$, Dimension $d = (n+1)^2 - 1$
- $\mathfrak{B}_n$: $O(2n+2)$, $d = n(2n+2)$
- $\mathfrak{C}_n$: $Usp(2n)$, $d = n(2n+1)$
- $\mathfrak{D}_n$: $O(2n)$, $d = n(2n-1)$
- Exceptional Lie algebras $\mathfrak{G}_2$, $\mathfrak{F}_4$, $\mathfrak{E}_6$, $\mathfrak{E}_7$, $\mathfrak{E}_8$

We will now have a closer look on $\mathfrak{A}_4$, since $SU(5)$ was our main concern in our analysis so far.

A.2. **The simple Lie algebra $\mathfrak{A}_4$.** The simple Lie algebra $\mathfrak{sl}(5,\mathbb{C})$ has rank $r = 4$. The elements of the Cartan subalgebra are then H^i with $i = 1, \dots, 4$. In an orthonormal basis $\{e_i\}$ one constructs the roots for $\mathfrak{A}_n$ according to $e_i - e_j$ with $i, j \in \{1, \dots, n+1\}$. The roots for $\mathfrak{A}_4$ are listed in table A.2. From there one sees that it is sufficient to work with the positive simple roots. There are four positive simple roots and six non-simple positive ones, leading to twenty non-vanishing roots. We denote those by $E^{\pm\alpha(i)} =: E^i_\pm$, $i = 1, \dots, 10$. There remains some freedom in the normalisation. The simplest normalisation is chosen in the so called Chevalley-Serre basis[23] where all structure constants are integers in the range of values ± 2.

[23] This basis was introduced by Chevalley in 1955 and led — after fifty years — to the discovery of new finite simple groups.

A.3. **Chevalley-Serre Basis.** Here are their Lie-bracket relations listed. The relations of the form $[H, E] = E$:

$$
\begin{array}{ccc}
[H^1, E^1_\pm] = \pm 2E^1_\pm & [H^4, E^9_\pm] = \mp 1E^{10}_\pm & [H^8, E^2_\pm] = \pm 1E^2_\pm \\
[H^1, E^2_\pm] = \mp 1E^2_\pm & [H^4, E^{10}_\pm] = \pm 1E^{10}_\pm & [H^8, E^3_\pm] = \pm 2E^3_\pm \\
[H^1, E^5_\pm] = \pm 1E^5_\pm & [H^5, E^1_\pm] = \pm 1E^1_\pm & [H^8, E^4_\pm] = \mp 1E^4_\pm \\
[H^1, E^6_\pm] = \pm 1E^6_\pm & [H^5, E^2_\pm] = \pm 1E^2_\pm & [H^8, E^6_\pm] = \pm 1E^6_\pm \\
[H^1, E^7_\pm] = \pm 1E^7_\pm & [H^5, E^3_\pm] = \mp 1E^3_\pm & [H^8, E^8_\pm] = \pm 2E^8_\pm \\
[H^1, E^8_\pm] = \mp 1E^8_\pm & [H^5, E^5_\pm] = \pm 2E^5_\pm & [H^8, E^9_\pm] = \pm 1E^9_\pm \\
[H^1, E^9_\pm] = \mp 1E^9_\pm & [H^5, E^6_\pm] = \mp 1E^6_\pm & [H^9, E^1_\mp] = \mp 1E^1_\pm \\
[H^2, E^1_\pm] = \mp 1E^1_\pm & [H^5, E^7_\pm] = \pm 1E^7_\pm & [H^9, E^2_\pm] = \pm 1E^2_\pm \\
[H^2, E^2_\pm] = \pm 2E^2_\pm & [H^5, E^9_\pm] = \pm 1E^9_\pm & [H^9, E^3_\pm] = \pm 1E^3_\pm \\
[H^2, E^3_\pm] = \mp 1E^3_\pm & [H^5, E^{10}_\pm] = \mp 1E^{10}_\pm & [H^9, E^4_\pm] = \pm 1E^4_\pm \\
[H^2, E^5_\pm] = \pm 1E^5_\pm & [H^6, E^1_\pm] = \pm 1E^1_\pm & [H^9, E^7_\pm] = \pm 1E^7_\pm \\
[H^2, E^8_\pm] = \pm 1E^8_\pm & [H^6, E^3_\pm] = \pm 2E^1_\pm & [H^9, E^8_\pm] = \pm 1E^8_\pm \\
[H^2, E^9_\pm] = \pm 1E^9_\pm & [H^6, E^4_\pm] = \mp 1E^4_\pm & [H^9, E^9_\pm] = \pm 2E^9_\pm \\
[H^2, E^{10}_\pm] = \mp 1E^{10}_\pm & [H^6, E^5_\pm] = \pm 1E^5_\pm & [H^9, E^{10}_\pm] = \pm 1E^{10}_\pm \\
[H^3, E^2_\pm] = \mp 1E^2_\pm & [H^6, E^6_\pm] = \pm 2E^6_\pm & [H^{10}, E^1_\pm] = \pm 1E^1_\pm \\
[H^3, E^3_\pm] = \pm 2E^3_\pm & [H^6, E^7_\pm] = \pm 1E^7_\pm & [H^{10}, E^2_\pm] = \mp 1E^2_\pm \\
[H^3, E^4_\pm] = \mp 1E^4_\pm & [H^6, E^8_\pm] = \pm 1E^8_\pm & [H^{10}, E^3_\pm] = \pm 1E^3_\pm \\
[H^3, E^5_\pm] = \mp 1E^5_\pm & [H^7, E^1_\pm] = \pm 1E^1_\pm & [H^{10}, E^4_\pm] = \pm 1E^4_\pm \\
[H^3, E^6_\pm] = \pm 1E^6_\pm & [H^7, E^3_\pm] = \pm 1E^3_\pm & [H^{10}, E^5_\pm] = \mp 1E^5_\pm \\
[H^3, E^8_\pm] = \pm 1E^8_\pm & [H^7, E^4_\pm] = \pm 1E^4_\pm & [H^{10}, E^7_\pm] = \pm 1E^7_\pm \\
[H^3, E^{10}_\pm] = \pm 1E^{10}_\pm & [H^7, E^5_\pm] = \pm 1E^5_\pm & [H^{10}, E^9_\pm] = \pm 1E^9_\pm \\
[H^4, E^3_\pm] = \mp 1E^3_\pm & [H^7, E^6_\pm] = \pm 1E^6_\pm & [H^{10}, E^{10}_\pm] = \pm 2E^{10}_\pm \\
[H^4, E^4_\pm] = \pm 2E^4_\pm & [H^7, E^7_\pm] = \pm 2E^7_\pm & \\
[H^4, E^6_\pm] = \mp 1E^6_\pm & [H^7, E^9_\pm] = \pm 1E^9_\pm & \\
[H^4, E^7_\pm] = \pm 1E^7_\pm & [H^7, E^{10}_\pm] = \pm 1E^{10}_\pm & \\
[H^4, E^8_\pm] = \mp 1E^8_\pm & [H^8, E^1_\pm] = \mp 1E^1_\pm &
\end{array}
$$

The relations of the form $[E, E] = E$:

$$
\begin{array}{cc}
[E^1_\pm, E^2_\pm] = \pm 1E^5_\pm & [E^3_\pm, E^4_\pm] = \pm 1E^{10}_\pm \\
[E^1_\mp, E^5_\pm] = \pm 1E^2_\pm & [E^3_\pm, E^5_\pm] = \pm 1E^6_\pm \\
[E^1_\mp, E^6_\pm] = \pm 1E^8_\pm & [E^3_\mp, E^6_\pm] = \pm 1E^5_\pm \\
[E^1_\mp, E^7_\pm] = \pm 1E^9_\pm & [E^3_\mp, E^8_\pm] = \pm 1E^2_\pm \\
[E^1_\pm, E^8_\pm] = \pm 1E^6_\pm & [E^3_\mp, E^{10}_\pm] = \pm 1E^4_\pm
\end{array}
$$

$$
\begin{array}{ll}
[E^1_\pm, E^9_\pm] = \pm 1 E^7_\pm & [E^4_\pm, E^6_\pm] = \pm 1 E^7_\pm \\
[E^2_\pm, E^3_\pm] = \pm 1 E^8_\pm & [E^4_\mp, E^7_\pm] = \pm 1 E^6_\pm \\
[E^2_\mp, E^5_\pm] = \pm 1 E^1_\pm & [E^4_\pm, E^8_\pm] = \pm 1 E^9_\pm \\
[E^2_\mp, E^8_\pm] = \pm 1 E^3_\pm & [E^4_\mp, E^9_\pm] = \pm 1 E^8_\pm \\
[E^2_\mp, E^9_\pm] = \pm 1 E^{10}_\pm & [E^4_\mp, E^{10}_\pm] = \pm 1 E^3_\pm \\
[E^2_\pm, E^{10}_\pm] = \pm 1 E^9_\pm &
\end{array}
$$

The relations of the form $[E_+, E_-] = H$:

$$
\begin{array}{ll}
[E^1_\pm, E^1_\mp] = \pm 1 H^1 & [E^6_\pm, E^6_\mp] = \pm 1 H^6 \\
[E^2_\pm, E^2_\mp] = \pm 1 H^2 & [E^7_\pm, E^7_\mp] = \pm 1 H^7 \\
[E^3_\pm, E^3_\mp] = \pm 1 H^3 & [E^8_\pm, E^8_\mp] = \pm 1 H^8 \\
[E^4_\pm, E^4_\mp] = \pm 1 H^4 & [E^9_\pm, E^9_\mp] = \pm 1 H^9 \\
[E^5_\pm, E^5_\mp] = \pm 1 H^5 & [E^{10}_\pm, E^{10}_\mp] = \pm 1 H^{10}
\end{array}
$$

The Killing form for those structure constants is according to [2], eq. (3.5.9)

$$\kappa_{ad} = \frac{1}{I_{\text{ad}}} \sum_{bc}^{20} f^b_{ac} f^c_{db}$$

where I_{ad} is a normalisation due to the adjoint representation. We get $\kappa(H^i, H^j) = I_{\text{ad}}\kappa^{ij}$ for the genenerators of the Cartan subalgebra and $\kappa(E^\alpha, E^\beta) = \frac{2}{(\alpha,\alpha)}\delta_{\alpha,-\beta}$. By means of the Killing form one can make the structure constants *totally* antisymmetric by defining

$$f_{abc} := f^n_{ab}\kappa_{nc} \tag{A.1}$$

where κ denotes the Killing form, which in matrix form looks like

$$\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_{n-p} \end{pmatrix} \tag{A.2}$$

where $\mathbb{1}_p$ is a $p \times p$ unit matrix, $p \geq 0$. A form is called *compact* if $p = 0$; the Killing form is then trivial. However there is another possibility of obtaining totally antisymmetric structure constants. By recombining the generators we can get hermitian operators leading directly to totally antisymmetric structure constants and a trivial Killing form (compact form).

A.4. **The compact form $\mathfrak{su}(5)$.** So far we have worked with $\mathfrak{sl}(5)$. Since we must achieve antisymmetric structure constants and hermitian generators, we go over to the compact form $\mathfrak{su}(5)$. We will therefore consider the following combination of the generators

$$\{iH^i | i = 1, \ldots, r\} \cup \{\frac{1}{2}\sqrt{(\alpha,\alpha)}(E^\alpha + E^{-\alpha}), \frac{i}{2}\sqrt{(\alpha,\alpha)}(E^\alpha - E^{-\alpha})\}$$

The Killing form is then trivial[24]

$$\kappa_{ab} = +\delta_{ab}$$

[24]In the case of anti-hermitian generators one gets a minus sign.

and one has directly totally antisymmetric structure constants. We denote our redefined generators by

$$H^i \longrightarrow iH^i \quad E^i := E^i_+ + E^i_- \quad E^i_\star := iE^i_+ - iE^i_-$$

The Lie-bracket relations (and structure constants) for the new generators follow directly from the relations in A.3. We will in the following relabel the generators with numbers $1, \dots, 24$ in order to meet with [29]. With these new generators we have thus found a basis for the real compact form $\mathfrak{su}(n)$. The complex algebra $\mathfrak{sl}(n, \mathbb{C})$ is often preferred in order to facilitate the algebraic handling of the roots; secular equations do in this case always find a solution since $\mathbb{C}$ is algebraically closed.

A.5. **Orthogonalisation.** For the diagonal elements a redefinition is possible. The normalisation is free. Most authors normalise in a way to obtain the same trace-formula as in the case of the non-diagonal elements[25]. This is the case for [29] and according to Weinberg [5] eq. (21.5.1)

$$\mathrm{tr}\,\{T_\alpha T_\beta\} = N_D \delta_{\alpha\beta}$$

where N_D depends on the representation and is here $N_D = 2$. One has now to proceed as in [3], ch. 3.5 for the case of $\mathfrak{su}(3)$, by redefining the Cartan subalgebra generators H_i and calculating the new commutation relations which yields the following 66 non-vanishing structure constants.

A.6. **Antisymmetric orthogonal structure constants.**

$$\begin{array}{rlrlrl}
f_{123} &= +1 & f_{4(16)(21)} &= +1/2 & f_{9(16)(23)} &= +1/2 \\
f_{147} &= +1/2 & f_{4(17)(20)} &= -1/2 & f_{9(17)(22)} &= -1/2 \\
f_{156} &= -1/2 & f_{59(13)} &= +1/2 & f_{(10)(16)(22)} &= +1/2 \\
f_{19(12)} &= +1/2 & f_{5(10)(14)} &= +1/2 & f_{(10)(17)(23)} &= +1/2 \\
f_{1(10)(11)} &= -1/2 & f_{5(16)(20)} &= +1/2 & f_{(11)(12)(15)} &= +2/\sqrt{6} \\
f_{1(16)(19)} &= +1/2 & f_{5(17)(21)} &= +1/2 & f_{(11)(18)(23)} &= +1/2 \\
f_{1(17)(18)} &= -1/2 & f_{678} &= +\sqrt{3}/2 & f_{(11)(19)(22)} &= -1/2 \\
f_{246} &= +1/2 & f_{6(11)(14)} &= +1/2 & f_{(12)(18)(22)} &= +1/2 \\
f_{257} &= +1/2 & f_{6(12)(13)} &= -1/2 & f_{(12)(19)(23)} &= +1/2 \\
f_{29(11)} &= +1/2 & f_{6(18)(21)} &= +1/2 & f_{(13)(14)(15)} &= +2/\sqrt{6} \\
f_{2(10)(12)} &= +1/2 & f_{6(19)(20)} &= -1/2 & f_{(13)(20)(23)} &= +1/2 \\
f_{2(16)(18)} &= +1/2 & f_{7(11)(13)} &= +1/2 & f_{(13)(21)(22)} &= -1/2 \\
f_{2(17)(19)} &= +1/2 & f_{7(12)(14)} &= +1/2 & f_{(14)(20)(22)} &= +1/2 \\
f_{345} &= +1/2 & f_{7(18)(20)} &= +1/2 & f_{(14)(21)(23)} &= +1/2 \\
f_{367} &= -1/2 & f_{7(19)(21)} &= +1/2 & f_{(15)(16)(17)} &= +1/2\sqrt{6} \\
f_{39(10)} &= +1/2 & f_{89(10)} &= +1/2\sqrt{3} & f_{(15)(18)(19)} &= +1/2\sqrt{6}
\end{array}$$

[25]One can also impose a normalisation of the roots as e.g.

$$\sum \alpha_j \alpha_j = \delta_{ij}.$$

In the case of $\mathfrak{su}(6)$ this is done in [9], ch. VIII.

$$
\begin{array}{rlrlrl}
f_{3(11)(12)} &= -1/2 & f_{8(11)(12)} &= +1/2\sqrt{3} & f_{(15)(20)(21)} &= +1/2\sqrt{6} \\
f_{3(16)(17)} &= +1/2 & f_{8(13)(14)} &= -1/\sqrt{3} & f_{(15)(22)(23)} &= -3/2\sqrt{6} \\
f_{3(18)(19)} &= -1/2 & f_{8(16)(17)} &= +1/2\sqrt{3} & f_{(16)(17)(24)} &= +5/2\sqrt{10} \\
f_{458} &= +\sqrt{3}/2 & f_{8(18)(19)} &= +1/2\sqrt{3} & f_{(18)(19)(24)} &= +5/2\sqrt{10} \\
f_{49(14)} &= +1/2 & f_{8(20)(21)} &= -1/\sqrt{3} & f_{(20)(21)(24)} &= +5/2\sqrt{10} \\
f_{4(10)(13)} &= -1/2 & f_{9(10)(15)} &= +2\sqrt{6} & f_{(22)(23)(24)} &= +5/2\sqrt{10}
\end{array}
$$

Note again that different normalisation for diagonal an off-diagonal elements is in principle possible; it is exactly the freedom we used for the orthogonalisation procedure.

For the generators we have the following correspondence: $X \in \{9, 10, 11, 12, 13, 14\}$, $Y \in \{16, 17, 18, 19, 20, 21\}$. Another choice is not possible from there. The gluon generators are assigned to $\lambda \in \{1, \ldots, 8\}$ and the electroweak generators are $\tau \in \{22, 23, 24\}$, with $24 = Z_0, \gamma = 15$.

These values agree with the literature and give us a check for our Mathematica package, calculating structure constants and Killing forms. We have here adapted the original program [26] for this case (cf. appendix B) and obtain the same values as in the literature [29].

A.7. **Gell-Mann matrices, structure constants for $\mathfrak{su}(3)$.** How are the structure constants calculated in conventional formalism? For $\mathfrak{su}(3)$ this can be done in the following way. One generalises the Pauli spin matrices for 3×3 matrices. Then the orthogonalisation is done, as we had to do it with the Cartan-Weyl basis. The structure constants follow from anti-commuting these matrices. Let's have a quick look on the properties of the Gell Mann matrices and their use in QCD.[26] In this way our identification for the gluon generators (not the gluon fields!) is made apparent. We know that quarks carry colour and we can write

$$q = \begin{pmatrix} q_r \\ q_w \\ q_b \end{pmatrix}, \; q \longrightarrow Uq \tag{A.3}$$

with a unitary matrix U

$$UU^{+} = 1, \; \det U = 1$$

that is normally written with a Hermitian matrix H in the form

$$U = e^{iH}, \; H = H^{+}, \; \operatorname{tr} H = 0. \tag{A.4}$$

A unitary matrix has $n^2 - 1$ parameters and so we have eight generators and we write

$$U = \exp\left[i \frac{\lambda_a}{2} \varepsilon_a\right] \tag{A.5}$$

with eight independent parameters ε_a which can be the "labels" for the gauge potential later. The generators are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.6}$$

[26]There is also a flavour $\mathfrak{su}(3)$ symmetry between quarks. Though it is not exact and we mean here throughout colour $\mathfrak{su}(3)$.

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{A.7}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A.8}$$

The structure constants of $\mathfrak{su}(3)$ that are simply obtained by commuting these matrices

$$[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}] = i f_{abc} \frac{\lambda_c}{2} \tag{A.9}$$

are:

$$\begin{aligned} &f_{123} = 1 \\ &f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = f_{367} = -f_{367} = 1/2 \\ &f_{458} = f_{678} = \sqrt{3}/2. \end{aligned} \tag{A.10}$$

For the gauge potential we have the form

$$A = \begin{pmatrix} A_3 + A_8/\sqrt{3} & A_2^1 & A_3^1 \\ A_1^2 & -A_3 + A_8/\sqrt{3} & A_3^2 \\ A_1^3 & A_2^3 & -2A_8/\sqrt{3} \end{pmatrix}. \tag{A.11}$$

Note that here the diagonal entries correspond just to the sum of the diagonal entries of the λ_3 and λ_8. There is no rotation done. The gluon generators $A_a^\mu(x)$ with an additional Lorentz index, which was oppressed so far, are defined as [7], eq. (12.2):

$$A^\mu(x) := \frac{1}{2}\lambda_a A_a^\mu(x). \tag{A.12}$$

The gluon field strengths are defined by the right hand side of

$$G^{\mu\nu} = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + ig[A^\mu(x), A^\nu(x)] \tag{A.13}$$

where g is a just a number, the unnormalised coupling constant. With (A.9) we rewrite this

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c \tag{A.14}$$

which is just a generalisation from $\mathfrak{su}(2)$ where instead of f_{abc}, ϵ_{123} has to be inserted. Like the QED Lagrangian the gluonic QCD Lagrangian is then

$$\mathcal{L} = -\frac{1}{2} \operatorname{tr} G_{\mu\nu} G^{\mu\nu} = -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}, \tag{A.15}$$

given by [20]

$$\mathcal{L} = -\frac{1}{4}[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c][\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{ab'c'} A_{b'}^\mu A_{c'}^\nu] \tag{A.16}$$

with a "free" part and an interacting part with the coupling constant g. The first order interaction part is

$$\begin{aligned} \mathcal{L}^{int} &= -\frac{1}{4}[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a][g f_{ab'c'} A_{b'}^\mu A_{c'}^\nu] - \frac{1}{4}[g f_{abc} A_\mu^b A_\nu^c][\partial^\mu A_a^\nu - \partial^\nu A_a^\mu] \\ &= -\frac{1}{2} g f_{abc} A_\mu^b A_\nu^c [\partial^\mu A_a^\nu - \partial^\nu A_a^\mu] \\ &= -\frac{1}{2} g f_{abc} A_\mu^a A_\nu^b [\partial^\mu A_c^\nu - \partial^\nu A_c^\mu] \\ &= g f_{abc} A_\mu^a A_\nu^b \partial A_c^\mu, \end{aligned} \tag{A.17}$$

which corresponds to our ansatz in eq. (1.13).

The procedure is generally used in physics: instead of starting from a Cartan-Weyl basis one sets off with generalised Gell-Mann matrices [10], calculates structure constants by commuting these matrices and gets the fields with the combinations above.

A.8. **Structure constants for the Georgi-Glashow $\mathfrak{su}(5)$ model with rotation.** We have calculated the structure constants for the Georgi-Glashow model described in section 7.1. with $\sin^2\theta_W = 3/8$ with a Mathematica-package. We notice that the number of structure constants does not change compared with appendix A.6. This is generally not true of course, with an arbitrary superposition of some of the (diagonal) generators the number of non-vanishing structure constants varies considerably.The assignment has changed and is listed in A.8. The diagonal generators take on the form

$$\begin{aligned}
\lambda_3 &= \mathrm{diag}(1,-1,0,0,0)\\
\lambda_8 &= \mathrm{diag}(1/\sqrt{3},1/\sqrt{3},-2/\sqrt{3},0,0)\\
Z &= \mathrm{diag}(1/\sqrt{10},1/\sqrt{10},1/\sqrt{10},1/\sqrt{10},-2\sqrt{2/5})\\
\gamma &= \mathrm{diag}(1/\sqrt{6},1/\sqrt{6},1/\sqrt{6},-1/\sqrt{3/2},0).
\end{aligned} \tag{A.18}$$

The other generators remain the same as in the case above

$$\begin{aligned}
f_{15(15)} &= +1 & f_{4(11)(21)} &= +1/2\sqrt{3} & f_{8(14)(20)} &= -1/2\\
f_{16(16)} &= +1/2 & f_{4(12)(22)} &= +1/2\sqrt{3} & f_{9(10)(17)} &= +1/2\\
f_{17(17)} &= -1/2 & f_{4(13)(23)} &= -1/2\sqrt{3} & f_{9(12)(24)} &= +1/2\\
f_{18(18)} &= -1/2 & f_{4(14)(24)} &= -1/2\sqrt{6} & f_{9(14)(22)} &= +1\\
f_{19(19)} &= -1/2 & f_{56(17)} &= +1/2 & f_{(10)(13)(24)} &= -1/2\\
f_{1(11)(21)} &= +1/2 & f_{57(16)} &= +1/2 & f_{(10)(14)(23)} &= +1/2\\
f_{1(12)(22)} &= -1/2 & f_{58(19)} &= +1/2 & f_{(11)(12)(15)} &= -1/2\\
f_{26(16)} &= +\sqrt{3} & f_{59(18)} &= +1/2 & f_{(11)(13)(16)} &= -1/2\\
f_{27(17)} &= +\sqrt{3} & f_{5(11)(22)} &= +1/2 & f_{(11)(14)(17)} &= -1/2\\
f_{28(18)} &= +1/\sqrt{3} & f_{5(12)(22)} &= -1/2 & f_{(12)(13)(17)} &= +1/2\\
f_{29(19)} &= +1/\sqrt{3} & f_{67(15)} &= -1/2 & f_{(12)(14)(19)} &= +1/2\\
f_{2(10)(20)} &= -1/\sqrt{3} & f_{68(20)} &= +1/2 & f_{(13)(14)(20)} &= +1/2\\
f_{2(11)(21)} &= +1/2\sqrt{3} & f_{6(10)(18)} &= -1/2 & f_{(15)(16)(17)} &= +1/2\\
f_{2(12)(22)} &= +1/2\sqrt{3} & f_{6(11)(23)} &= -1/2 & f_{(15)(18)(19)} &= -1/2\\
f_{2(13)(23)} &= -1/\sqrt{3} & f_{6(13)(20)} &= -1/2 & f_{(15)(21)(22)} &= +1/2\\
f_{3(11)(21)} &= +1/2\sqrt{10} & f_{79(19)} &= +1/2 & f_{(16)(18)(20)} &= -1/2\\
f_{3(12)(22)} &= +1/2\sqrt{10} & f_{7(10)(18)} &= +1/2 & f_{(16)(21)(23)} &= -1/2\\
f_{3(13)(23)} &= -1/2\sqrt{10} & f_{7(12)(23)} &= +1/2 & f_{(17)(19)(20)} &= +1/2\\
f_{3(14)(24)} &= +1/2\sqrt{10} & f_{7(13)(22)} &= -1/2 & f_{(17)(22)(23)} &= +1/2
\end{aligned}$$

Table 6. Assignment for generators in Georgi-Glashow $\mathfrak{su}(5)$.

$1 = \lambda^3$	$7 = \lambda^6$	$13 = Y^5$	$19 = X^4$
$2 = \lambda^8$	$8 = X^1$	$14 = W^1$	$20 = X^6$
$3 = Z$	$9 = X_3$	$15 = \lambda_2$	$21 = Y^2$
$4 = \gamma$	$10 = X_5$	$16 = \lambda_5$	$22 = Y^4$
$5 = \lambda_1$	$11 = Y^1$	$17 = \lambda_7$	$23 = Y^6$
$6 = \lambda_4$	$12 = Y^3$	$18 = X^2$	$24 = W^2$

$$
\begin{aligned}
f_{48(18)} &= +\sqrt{2/3} & f_{89(15)} &= +1/2 & f_{(18)(21)(24)} &= -1/2 \\
f_{49(19)} &= +\sqrt{2/3} & f_{8(10)(16)} &= +1/2 & f_{(19)(22)(24)} &= +1/2 \\
f_{4(10)(20)} &= +\sqrt{2/3} & f_{8(11)(24)} &= +1/2 & f_{(20)(23)(24)} &= -1/2
\end{aligned}
$$

One can also try other superpositions of the diagonal generators. For instance we could have defined our the B field differently, being already a sum of the diagonal matrices of the adjoint representation. This gives a different coupling structure again.

Appendix B. Program code

In order to obtain concrete values for the structure constants we worked with a Mathematica package [26][27]. We added the function *createGGbasis* similar to the calculation of an orthonormal basis

```
createONBasissuN[n_] := Block[{i,r,p,B,A,a},
r = n-1; (*Rank of su(n)*)
m = n^2 -1; (*Dimension of su(n)*)
p = 1/2  n(n-1);
A = Table[a[i], {i,1,m}];
B = createTriangularBasisslN[n];
For[i=1, i<=r, i++,
a[i] = B[[i]];
];
For[i=1, i<=p, i++,
a[i+r] = (B[[r+i]] + B[[r+p+i]]);
a[i+r+p] = -I (B[[r+i]] - B[[r+p+i]]);
];
A]
```

In the following function one calculates A and B as before. One has here to provide the program with the diagonal generators when they differ from those that are produced by the program. By means of W^3 and B^3 which are given in the Georgi-Glashow model we can rotate them in order to obtain the Z and the A^μ as in ordinary $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

```
createGGbasis[n_] := Block[{i,r,p,B,A,a,G1,G2,g,m,Z,Ph,theta},
(*Model with normalisation according to Georgi Glashow (GG)*)
Clear[A,B,a];
```

[27]The package can be downloaded from http://library.wolfram.com/infocenter/MathSource/4806/. In the meantime several resources available on the www have been elaborated, among them “LiE”: http://Young.sp2mi.univ-poitiers.fr/∼marc/LiE/ and “simplLie”: http://www.crm.umontreal.ca/∼rand/simpLie.html.

```
Unset[A];
Unset[a];
r = n-1;
g = r-2;
theta = ArcSin[Sqrt[3/8]];
m = n^2 -1; (*dimension for the Lie algebra su(n)*)
p = 1/2  n(n-1);
A = Table[a[i], {i,1,m}];
B = createTriangularBasisslN[n];
W3 = {0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
      1/Sqrt[2],0,0,0,0,0,-1/Sqrt[2]};
B3 = {2/Sqrt[30],0,0,0,0,0,2/Sqrt[30],0,0,0,0,0,
      2/Sqrt[30],0,0,0,0,0,-3/Sqrt[30],0,0,0,0,0,-3/Sqrt[30]};
Z = Cos[theta] W3  + Sin[theta] B3;
Ph = -Sin[theta] W3 + Cos[theta] B3;
For[i=1, i<=g, i++,
a[i] = Sqrt[2] B[[i]];
a[3] = Sqrt[2] Z;
a[4] = Sqrt[2] Ph;
  ];
  For[i=1, i<=p, i++,
a[i+r] = (B[[r+i]] + B[[r+p+i]]);
a[i+r+p] = I (B[[r+i]] - B[[r+p+i]]);
];
A]
```

Before running this function one checks whether the basis is really orthonormal. This can be done with the function *isBasisOrthonormal* and is true for $\sin^2(\theta) = 3/8$. The program is only valid for real compact Lie algebras.

Appendix C. Mathematical restrictions from the equation sets

C.1. **First solution from off-diagonal restrictions.** Since $f[1] = f[2] = f[3] = f[4] = 0$ we start in f with the index 5. In f we assume only one coupling which will be motivated below. Also in g we assume only one coupling to be different from zero; this is the index 6. In h though we assume two couplings $h[7]$ and $h[8]$. The indices $8, 9, 10$ are reserved for couplings to massless particles. Here always one index suffices since the mass factors factorise. Note that there are also other restrictions possible; one can also add to g or to f an additional coupling, but we discard such a possibility, since later in the physical discussion of the solutions it turns out that an additional coupling to f is not physical. It is of no importance whether one adds a coupling to g or to h, both lead formally to the same solutions.

$$
\begin{aligned}
f[5] := m[5]\big(&m[6]m[7]m[8]m[1] + m[2]m[1]m[6]^2 - m[2]^2m[1]^2 + m[2]^2m[1]m[3] \\
&+ 2m[8]m[6]m[2]m[1] - m[1]m[2]m[3]^2 - m[4]m[2]m[6]^2 - 2m[8]m[6]m[3]m[1] \\
&- m[4]^2m[2]m[1] + m[1]^2m[2]m[4] + m[4]^2m[2]m[6] + m[4]m[2]^2m[1] \\
&- m[3]m[1]m[6]^2 - 2m[3]m[4]m[1]m[2] - m[1]m[2]^2m[6] + m[4]m[2]^2m[6] \\
&+ m[3]^2m[1]m[6] - m[3]^2m[4]m[6] - m[4]m[3]m[1]^2 + m[4]m[3]^2m[1] \\
&+ m[3]m[4]^2m[1] - m[3]m[4]^2m[6] + m[3]m[4]m[6]^2 + m[3]m[4]^2m[2] \\
&- m[3]m[4]m[2]^2 + m[3]^2m[4]m[2] - m[3]^2m[4]^2 + m[6]m[7]m[8]m[2]
\end{aligned}
$$

$$
\begin{aligned}
&- 2m[4]m[8]m[6]m[2] + 2m[4]m[8]m[6]m[3] + m[3]m[1]^2m[2] + m[3]m[1]^2m[6] \\
&- m[7]m[1]m[8]m[2] + m[7]m[8]m[3]m[2] - 2m[7]m[3]m[6]m[1] + m[7]m[8]m[6]m[3] \\
&+ 2m[7]m[2]m[6]m[1] - 3m[7]m[8]m[6]^2 + m[4]m[7]m[1]m[8] - m[3]m[7]m[4]m[8] \\
&- 2m[4]m[7]m[2]m[6] - m[2]m[1]^2m[6] + 2m[4]m[7]m[3]m[6] + m[4]m[7]m[8]m[6]\big) \Big/ \\
&\big[m[6]\big(- m[3]^2m[4]m[5] - m[1]m[2]^2m[5] + m[4]m[2]^2m[5] + m[3]^2m[1]m[5] \\
&+ m[2]^2m[1]m[3] - m[1]m[2]m[3]^2 - m[4]^2m[2]m[1] + m[1]^2m[2]m[4] - m[4]m[2]^2m[1] \\
&+ m[3]^2m[1]^2 - m[3]m[1]m[5]^2 + 2m[3]m[4]m[1]m[2] + m[4]^2m[2]m[5] \\
&- m[4]m[3]m[1]^2 - m[4]m[3]^2m[1] + m[3]m[4]^2m[1] - m[3]m[4]^2m[2] \\
&- m[3]m[4]m[2]^2 + m[3]^2m[4]m[2] - m[4]m[2]m[5]^2 + m[4]m[3]m[5]^2 \\
&- m[3]m[4]^2m[5] + 2m[8]m[2]m[5]m[1] - 2m[8]m[3]m[5]m[1] - m[3]m[1]^2m[2] \\
&+ m[2]m[1]m[5]^2 + m[4]^2m[2]^2 + m[3]m[1]^2m[5] + m[7]m[1]m[8]m[3] \\
&- 2m[4]m[8]m[2]m[5] + 2m[4]m[8]m[3]m[5] - m[7]m[8]m[3]m[2] - m[7]m[8]m[5]m[3] \\
&- m[7]m[8]m[5]m[2] + 2m[7]m[2]m[5]m[1] - 2m[7]m[3]m[5]m[1] - m[4]m[7]m[8]m[5] \\
&- 2m[4]m[7]m[2]m[5] + 2m[4]m[7]m[3]m[5] + m[4]m[7]m[8]m[2] - m[4]m[7]m[1]m[8] \\
&+ 3m[7]m[8]m[5]^2 - m[7]m[8]m[1]m[5] - m[2]m[1]^2m[5]\big)\big]
\end{aligned}
\tag{C.1}
$$

Note $f[5]$ can here also stand for $\sum_i f_{X^1X^2i}f_{X^3X^4i}$ with at least three different masses for X^1, X^2, X^3 and X^4 (cf. (13.1)), if such couplings exist in the theory. That is why we do not yet specify for W^1 and W^2 in the product and the masses. We do this only when discussing physically relevant restrictions.

$$
g[6] = 0, \tag{C.2}
$$

$$
\begin{aligned}
h[7] := m[7]\big(&m[4]m[2]^2m[6]m[3] - m[4]m[2]m[3]^2m[6] + m[1]m[2]m[3]^2m[6] \\
&- m[4]^2m[2]^2m[6] - m[1]m[2]^2m[6]m[3] + m[3]m[1]^2m[6]m[2] + m[3]m[1]^2m[5]m[2] \\
&+ m[1]m[2]^2m[3]m[5] - m[1]^2m[2]^2m[5] - m[4]m[2]^2m[3]m[5] + m[4]m[2]m[3]^2m[5] \\
&- m[1]m[2]m[3]^2m[5] - m[3]^2m[1]^2m[6] - m[3]m[1]m[6]m[4]^2 + m[1]m[2]m[6]m[4]^2 \\
&- m[1]m[2]m[4]^2m[5] + m[3]^2m[1]m[4]m[6] + m[4]m[2]^2m[1]m[6] \\
&+ m[3]m[1]m[4]^2m[5] - 2m[4]m[1]m[6]m[3]m[2] + m[4]^2m[6]m[3]m[2] \\
&+ 2m[8]m[1]m[6]m[3]m[5] + 2m[8]m[4]m[6]m[5]m[2] + m[4]m[2]m[6]m[5]^2 \\
&+ 2m[8]m[4]m[6]m[3]m[5] - m[4]m[3]m[6]m[5]^2 - m[1]m[2]m[6]m[5]^2 \\
&+ m[3]m[1]m[6]m[5]^2 + m[4]^2m[5]m[3]m[2] - 2m[4]m[1]m[5]m[3]m[2] \\
&- m[4]^2m[5]m[3]^2 + m[4]m[3]^2m[1]m[5] + m[4]m[3]m[5]m[6]^2 - m[4]m[2]m[5]m[6]^2 \\
&+ m[4]m[5]m[1]m[2]^2 + m[1]m[2]m[5]m[6]^2 - m[3]m[1]m[5]m[6]^2 \\
&+ 2m[8]m[1]m[6]m[5]m[2] + m[3]m[1]^2m[6]m[4] - m[1]^2m[2]m[6]m[4] \\
&- m[1]^2m[3]m[4]m[5] + m[4]m[1]^2m[2]m[5] - m[4]^2m[1]^2m[8] - m[2]^2m[3]^2m[8] \\
&+ 3m[5]^2m[8]m[6]^2 + m[2]m[3]m[8]m[6]^2 + 3m[8]^2m[6]m[5]^2 + m[2]^2m[3]m[1]m[8] \\
&- 2m[3]m[5]m[8]m[6]^2 + m[5]m[6]m[8]m[4]^2 + 3m[8]^2m[5]m[6]^2 + m[4]m[1]m[8]m[6]^2 \\
&+ m[5]m[8]m[1]^2m[6] - 2m[4]m[6]m[8]m[5]^2 - 2m[1]m[6]m[8]m[5]^2 \\
&+ 2m[5]m[2]m[8]m[6]m[3] - m[5]^2m[1]m[8]m[2] + 2m[1]m[6]m[8]m[4]m[5] \\
&- 2m[2]m[3]m[4]m[1]m[8] - 2m[5]^2m[6]m[8]m[2] - m[5]^2m[3]m[4]m[8]
\end{aligned}
$$

$$
\begin{aligned}
&+ m[8]^2m[6]m[3]m[1] - m[4]m[2]m[8]m[6]^2 + m[4]^2m[2]m[1]m[8] - m[4]m[2]^2m[1]m[8]\\
&+ m[4]m[2]^2m[8]m[3] + m[4]m[1]^2m[8]m[2] + m[3]^2m[5]m[8]m[6] - m[3]^2m[1]m[4]m[8]\\
&- 2m[8]^2m[6]m[1]m[5] - m[8]^2m[5]m[3]m[2] - m[8]^2m[6]m[3]m[2] + m[3]m[1]^2m[4]m[8]\\
&- 2m[8]^2m[6]m[4]m[5] - 2m[8]^2m[6]m[5]m[2] + m[8]^2m[6]m[4]m[2] + m[8]^2m[5]m[1]m[2]\\
&+ m[8]^2m[5]m[3]m[4] - m[8]^2m[5]m[4]m[1] - m[3]m[1]m[8]m[6]^2 - m[8]^2m[6]m[4]m[1]\\
&+ m[3]^2m[1]m[8]m[2] - 2m[4]m[8]m[5]m[6]^2 - 2m[8]^2m[6]m[3]m[5] - 2m[5]m[2]m[8]m[6]^2\\
&- 2m[1]m[5]m[8]m[6]^2 - m[3]m[1]^2m[8]m[2] + m[5]m[2]^2m[6]m[8] + m[2]m[3]^2m[4]m[8]\\
&+ m[5]^2m[4]m[1]m[8] + m[4]^2m[1]m[3]m[8] - 2m[5]^2m[8]m[6]m[3] + m[5]^2m[8]m[3]m[2]\\
&- m[4]^2m[2]m[3]m[8]\Big) \Big/ m[6]\big(- m[3]^2m[4]m[5] - m[1]m[2]^2m[5] + m[4]m[2]^2m[5]\\
&+ m[3]^2m[1]m[5] + m[2]^2m[1]m[3] - m[1]m[2]m[3]^2 - m[4]^2m[2]m[1] + m[1]^2m[2]m[4]\\
&- m[4]m[2]^2m[1] + m[3]^2m[1]^2 - m[3]m[1]m[5]^2 + 2m[3]m[4]m[1]m[2] + m[4]^2m[2]m[5]\\
&- m[4]m[3]m[1]^2 - m[4]m[3]^2m[1] + m[3]m[4]^2m[1] - m[3]m[4]^2m[2] - m[3]m[4]m[2]^2\\
&+ m[3]^2m[4]m[2] - m[4]m[2]m[5]^2 + m[4]m[3]m[5]^2 - m[3]m[4]^2m[5] + 2m[8]m[2]m[5]m[1]\\
&- 2m[8]m[3]m[5]m[1] - m[3]m[1]^2m[2] + m[2]m[1]m[5]^2 + m[4]^2m[2]^2 + m[3]m[1]^2m[5]\\
&+ m[7]m[1]m[8]m[3] - 2m[4]m[8]m[2]m[5] + 2m[4]m[8]m[3]m[5] - m[7]m[8]m[3]m[2]\\
&- m[7]m[8]m[5]m[3] - m[7]m[8]m[5]m[2] + 2m[7]m[2]m[5]m[1] - 2m[7]m[3]m[5]m[1]\\
&- m[4]m[7]m[8]m[5] - 2m[4]m[7]m[2]m[5] + 2m[4]m[7]m[3]m[5] + m[4]m[7]m[8]m[2]\\
&- m[4]m[7]m[1]m[8] + 3m[7]m[8]m[5]^2 - m[7]m[8]m[1]m[5] - m[2]m[1]^2m[5]\big)\big(m[7] - m[8]\big)
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
h[8] := {}&(m[4]m[2]^2m[6]m[3] - m[4]m[2]m[3]^2m[6] + m[1]m[2]m[3]^2m[6] - m[4]^2m[2]^2m[6]\\
&- m[1]m[2]^2m[6]m[3] + m[3]m[1]^2m[6]m[2] + m[3]m[1]^2m[5]m[2] + m[1]m[2]^2m[3]m[5]\\
&- m[1]^2m[2]^2m[5] - m[4]m[2]^2m[3]m[5] + m[4]m[2]m[3]^2m[5] - m[1]m[2]m[3]^2m[5]\\
&- m[3]^2m[1]^2m[6] - m[3]m[1]m[6]m[4]^2 + m[1]m[2]m[6]m[4]^2 - m[1]m[2]m[4]^2m[5]\\
&+ m[3]^2m[1]m[4]m[6] + m[4]m[2]^2m[1]m[6] + m[3]m[1]m[4]^2m[5] + m[7]m[2]^2m[1]m[3]\\
&+ m[7]m[1]m[2]m[3]^2 + m[7]m[4]m[1]m[6]^2 - m[7]m[4]^2m[1]^2 - m[7]m[3]^2m[2]^2\\
&+ 3m[7]m[5]^2m[6]^2 - 2m[7]^2m[6]m[3]m[5] - 2m[7]^2m[6]m[1]m[5] - m[7]^2m[5]m[3]m[2]\\
&- 2m[7]^2m[6]m[5]m[2] - m[7]^2m[6]m[3]m[2] + m[7]^2m[5]m[1]m[2] - 2m[7]^2m[6]m[4]m[5]\\
&+ m[7]^2m[5]m[3]m[4] + m[7]^2m[6]m[4]m[2] - m[7]^2m[5]m[4]m[1] + 3m[7]^2m[5]m[6]^2\\
&- 2m[4]m[1]m[6]m[3]m[2] + m[4]^2m[6]m[3]m[2] + 2m[7]m[4]m[5]m[6]m[1]\\
&- m[7]^2m[6]m[4]m[1] + m[7]m[2]^2m[6]m[5] + m[7]m[3]^2m[6]m[5] + 2m[7]m[6]m[5]m[2]m[3]\\
&+ 3m[7]^2m[6]m[5]^2 - 2m[7]m[2]m[6]m[5]^2 - 2m[7]m[3]m[6]m[5]^2 + m[7]m[4]m[1]m[5]^2\\
&- 2m[7]m[3]m[5]m[6]^2 - 2m[7]m[2]m[5]m[6]^2 + m[7]m[1]^2m[5]m[6] - 2m[7]m[5]^2m[4]m[6]\\
&+ m[7]m[4]^2m[5]m[6] - 2m[7]m[5]^2m[6]m[1] + m[4]m[2]m[6]m[5]^2 - m[7]m[2]m[1]m[5]^2\\
&- 2m[7]m[4]m[5]m[6]^2 + 2m[7]m[4]m[6]m[3]m[5] - m[7]m[4]m[3]m[5]^2 - m[7]m[3]m[1]^2m[2]\\
&+ 2m[7]m[1]m[6]m[3]m[5] + 2m[7]m[1]m[6]m[5]m[2] + 2m[7]m[4]m[6]m[5]m[2]\\
&- m[4]m[3]m[6]m[5]^2 + m[7]m[3]^2m[4]m[2] + m[7]m[3]m[5]^2m[2] - m[1]m[2]m[6]m[5]^2\\
&+ m[3]m[1]m[6]m[5]^2 - m[7]m[3]m[4]^2m[2] + m[7]m[3]m[4]m[2]^2 - m[7]m[4]m[3]^2m[1]
\end{aligned}
$$

$$+ m[7]m[3]m[4]^2m[1] - 2m[7]m[3]m[4]m[1]m[2] + m[7]m[2]m[6]^2m[3] + m[7]m[4]m[3]m[1]^2$$
$$- m[7]m[4]m[2]^2m[1] - m[7]m[3]m[1]m[6]^2 - m[7]m[4]m[2]m[6]^2 + m[7]m[1]^2m[2]m[4]$$
$$+ m[7]m[4]^2m[2]m[1] + m[4]^2m[5]m[3]m[2] - 2m[4]m[1]m[5]m[3]m[2] - m[4]^2m[5]m[3]^2$$
$$+ m[4]m[3]^2m[1]m[5] + m[4]m[3]m[5]m[6]^2 - m[4]m[2]m[5]m[6]^2 + m[4]m[5]m[1]m[2]^2$$
$$+ m[1]m[2]m[5]m[6]^2 - m[3]m[1]m[5]m[6]^2 + m[7]^2m[6]m[3]m[1] + m[3]m[1]^2m[6]m[4]$$
$$-m[1]^2m[2]m[6]m[4] - 2m[7]m[1]m[5]m[6]^2 - m[1]^2m[3]m[4]m[5] + m[4]m[1]^2m[2]m[5]\big)m[8] \Big/$$
$$\big[m[6]\big(- m[3]^2m[4]m[5] - m[1]m[2]^2m[5] + m[4]m[2]^2m[5] + m[3]^2m[1]m[5] + m[2]^2m[1]m[3]$$
$$- m[1]m[2]m[3]^2 - m[4]^2m[2]m[1] + m[1]^2m[2]m[4] - m[4]m[2]^2m[1] + m[3]^2m[1]^2$$
$$- m[3]m[1]m[5]^2 + 2m[3]m[4]m[1]m[2] + m[4]^2m[2]m[5] - m[4]m[3]m[1]^2$$
$$- m[4]m[3]^2m[1] + m[3]m[4]^2m[1] - m[3]m[4]^2m[2] - m[3]m[4]m[2]^2 + m[3]^2m[4]m[2]$$
$$-m[4]m[2]m[5]^2 + m[4]m[3]m[5]^2 - m[3]m[4]^2m[5] + 2m[8]m[2]m[5]m[1] - 2m[8]m[3]m[5]m[1]$$
$$- m[3]m[1]^2m[2] + m[2]m[1]m[5]^2 + m[4]^2m[2]^2 + m[3]m[1]^2m[5] + m[7]m[1]m[8]m[3]$$
$$- 2m[4]m[8]m[2]m[5] + 2m[4]m[8]m[3]m[5] - m[7]m[8]m[3]m[2] - m[7]m[8]m[5]m[3]$$
$$- m[7]m[8]m[5]m[2] + 2m[7]m[2]m[5]m[1] - 2m[7]m[3]m[5]m[1] - m[4]m[7]m[8]m[5]$$
$$- 2m[4]m[7]m[2]m[5] + 2m[4]m[7]m[3]m[5] + m[4]m[7]m[8]m[2] - m[4]m[7]m[1]m[8]$$
$$+ 3m[7]m[8]m[5]^2 - m[7]m[8]m[1]m[5] - m[2]m[1]^2m[5]\big)\big(m[7] - m[8]\big)\big] \quad \text{(C.4)}$$
$$f[8] = g[9] = h[10] = 0. \quad \text{(C.5)}$$

The Jacobi-identity is fulfilled:

$$f[5] - g[6] + h[7] + h[8] = 0. \quad \text{(C.6)}$$

If one puts in mass degeneracies here assuming more than three masses in the theory, the rank of the matrix does not change, so this solution remains valid if we set $m[1] = m[3]$ or $m[2] = m[3]$ or $m[1] = m[4]$ or $m[4] = m[4]$. The rank does also not change when we discuss mass degeneracies for the new masses $m[5], \dots m[8]$ which we will do in section 13.1 below. In one case only, when $m[1] = m[2]$ *and* $m[3] = m[4]$ the rank decreases by one. We consider this solution separately in the next section.

Remark: One cannot get mass relations from

$$f[1] = f_{121}f_{341} = 0,$$

due to antisymmetry because the mass m_1 is not necessarily linked to a single particle, there can be a mass degeneracy.

C.2. **Second solution with mass degeneracy.** A mass degeneracy $m[1] = m[2]$ and $m[3] = m[4]$ does physically imply a different structure for the solution. We know from gauge invariance that in such a case the couplings to massless gauge bosons is in general not zero. We would thus suspect that the sum of products $\sum f_{12d}f_{34d}$ is not zero. This is indeed the case. Since the rank of the equation system is only five, we assume one summand less in h. In this way we are able to give restrictions on f and h, normalising g to one and we get a restriction for $\sum f_{12d}f_{34d}$. The new products of structure constants we denote with q, r and s.

$$q[5] := -(6m[1]^2m[4]^2 - 4m[1]^3m[4] - 4m[1]m[4]^3 + m[4]^4 - m[7]^2m[1]^2$$
$$+ 2m[1]m[4]m[6]^2 - m[4]^2m[6]^2 - m[1]^2m[6]^2 + m[1]^4 + 4m[7]^2m[1]m[6]$$
$$- 3m[7]^2m[6]^2 - 4m[7]m[1]^2m[6] + 4m[1]m[7]m[6]^2 + 4m[7]^2m[4]m[6]$$

$$+ 2m[7]^2 m[4]m[1] - m[7]^2 m[4]^2 - 4m[7]m[4]^2 m[6] + 4m[7]m[4]m[6]^2$$
$$- 8m[7]m[4]m[6]m[1]\big)\big(m[6]m[5](-3m[7]^2 + 2m[7]m[4] + 2m[1]m[7]$$
$$- 2m[1]m[4] + m[1]^2 + m[4]^2)\big) \quad \text{(C.7)}$$

$$r[6] := 1 \quad \text{(C.8)}$$

$$s[7] \ := \ -\frac{m[7](-3m[6]^2 + 2m[6]m[1] + m[4]^2 - 2m[1]m[4] + m[1]^2 + 2m[6]m[4])}{m[6](-3m[7]^2 + 2m[7]m[4] + 2m[1]m[7] - 2m[1]m[4] + m[1]^2 + m[4]^2)} \quad \text{(C.9)}$$

$$q[8] := (4m[7]m[4]m[6]^2 - 4m[1]m[4]^3 + 6m[1]^2 m[4]^2 + m[1]^4$$
$$+ 4m[7]^2 m[1]m[6] - 3m[7]^2 m[6]^2 + m[5]m[7]m[4]^2 + 2m[7]^2 m[4]m[1]$$
$$- m[7]^2 m[1]^2 + m[5]m[7]m[1]^2 - 2m[5]m[7]m[4]m[1] - 3m[5]m[7]m[6]^2$$
$$+ 2m[1]m[4]m[6]^2 - m[1]^2 m[6]^2 - m[7]^2 m[4]^2 - 4m[7]m[1]^2 m[6]$$
$$+ m[4]^4 - 4m[1]^3 m[4] + 4m[7]^2 m[4]m[6] - 3m[5]m[6]m[7]^2$$
$$+ m[5]m[6]m[4]^2 + 4m[4]m[5]m[6]m[7] - 4m[7]m[4]^2 m[6]$$
$$+ m[5]m[6]m[1]^2 + 4m[1]m[5]m[6]m[7] - 8m[7]m[4]m[6]m[1]$$
$$- 2m[5]m[6]m[1]m[4] + 4m[1]m[7]m[6]^2 - m[4]^2 m[6]^2)\big/(m[6]m[5]\times$$
$$(-3m[7]^2 + 2m[7]m[4] + 2m[1]m[7] - 2m[1]m[4] + m[1]^2 + m[4]^2)). \quad \text{(C.10)}$$

The Jacobi-identity is fulfilled

$$q[5] - r[6] + s[7] + q[8] = 0. \quad \text{(C.11)}$$

C.3. **General solution from semi-diagonal restrictions.** As already known from smaller theories [26] the gauge restrictions with one common index are very restrictive. In analysing all such relations we found though that the rank of the corresponding equation set is never bigger than three which means that we can provide restrictions on 3 sums of products of structure constants (6 non-vanishing couplings), setting one structure constant to one. These products of structure constants are not the same as those listed above. Again one is led to general solutions. Since we did not refer to them explicitly in the text, we do not list them here.

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